

Koszul Calculus for N -homogeneous algebras

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Abstract

We extend the Koszul calculus defined on quadratic algebras by Berger, Lambre, Solotar (Koszul calculus, arXiv:1512.00183) to N -homogeneous algebras for any integer $N \geq 2$, the quadratic algebras corresponding to $N = 2$. Koszul cup and cap products are introduced and are reduced to usual cup and cap products if $N = 2$, but if $N > 2$, they are defined by very specific expressions. These specific expressions are compatible with the Koszul differentials and provide associative products *on classes*. Actually, there is no associativity in general on chains-cochains, suggesting that Koszul cochains should constitute an A_∞ -algebra, acting as an A_∞ -bimodule on Koszul chains.

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1 Introduction

In [9], the Koszul calculus was introduced as a new homological tool for studying quadratic algebras. The Koszul calculus consists in Koszul (co)homology, together with Koszul cup and cap products. If the quadratic algebra A is Koszul, the Koszul calculus is reduced to Hochschild calculus, but if A is not Koszul, the Koszul calculus is a new invariant of Manin's category. An application of the Koszul calculus to the Koszul duality is given in [9], showing that the true nature of Koszul duality Theorem does not depend on any assumption of Koszulity on A .

Comparing the Koszul calculus to the Tamarkin-Tsygan calculus [31], the main feature of the Koszul calculus is the fundamental formula

$$b_K(f) = -[e_A, f]_{\smile_K} \quad (1.1)$$

where b_K is the Koszul differential, e_A is the fundamental 1-cocycle, f is a Koszul cochain, and \smile_K denotes the Koszul cup product. The Koszul calculus is simpler since no Gerstenhaber bracket is needed in this formula. The Koszul calculus is more flexible and more symmetric since Formula (1.1) holds for any A -bimodule and exits in homology. Moreover, the higher Koszul calculus defined in [9] reveals some analogues of classical results of the Tamarkin-Tsygan calculus, providing a new information about the quadratic algebra A when A is not Koszul.

Motivated by cubic Artin-Schelter algebras [2], the author defined in [4] a notion of Koszulity for N -homogeneous algebras. This notion generalizes Koszul algebras defined by

Priddy and corresponding to $N = 2$ [30]. Since then, N -homogeneous algebras and N -Koszul algebras have been connected to the following domains.

1. Representation theory: a PBW theorem was obtained for N -Koszul algebras, including applications to higher symplectic reflection algebras [8]. This theorem was also applied to Calabi-Yau quiver algebras [11].
2. Theoretical physics: Yang-Mills algebras are Koszul cubic algebras [14, 15], and their PBW deformations were determined [6]. Other cubic algebras linked to parastatistics were studied [19].
3. Poincaré duality in Hochschild (co)homology (Van den Bergh duality [32]): this duality was applied to AS-Gorenstein N -Koszul algebras [10], and was studied in terms of preregular multilinear forms [17, 18] or twisted potentials [12].
4. N -complexes in homological algebra: N -Koszulity is linked to N -complexes by generalizing Manin's approach of quadratic algebras [29] to N -homogeneous algebras [7].
5. A_∞ -algebras: the Yoneda algebra of an N -Koszul algebra has an explicit structure of A_∞ -algebra and this structure gives a characterization of N -Koszulity [24, 28].
6. Quiver algebras: N -Koszulity was extended to quiver algebras with relations [21].
7. Combinatorics: a MacMahon Master Theorem was proved for any N -Koszul algebra [23]. Moreover, the link with the combinatorics of distributive lattices was studied in [5], and including certain monoidal categories, in [25].
8. Confluence and rewriting systems: the confluence of N -Koszul algebras implies an explicit contracting homotopy of the Koszul resolution [13]. In general, the confluence is well understood in terms of higher dimensional linear rewriting systems [22].
9. Operads: the Koszul duality for operads is known to be essentially quadratic [27]. Recently, an extension to the N -case was proposed in [16].

The aim of the paper is to extend the Koszul calculus of quadratic algebras to N -homogeneous algebras for any $N \geq 2$. As for $N = 2$, the extended Koszul calculus provides homological invariants independently of any assumption of Koszulity. These invariants form a graded associative algebra on the cohomological side, acting as a graded bimodule on the homological side. We think that the Koszul calculus of N -homogeneous algebras will bring new advances for the subjects developed in the different items listed above.

Let us describe the contents of the paper. Koszul homology and Koszul cohomology of N -homogeneous algebras A are defined in Section 2. The bimodule complex $K(A)$ was already known, but it was used up to now only when A is Koszul. The idea is to use $K(A)$ in order to obtain invariants even if A is not Koszul, and these invariants may be different from Hochschild classes. Koszul (co)homology defines a δ -functor and is isomorphic to a Hochschild hyper(co)homology. Moreover, the invariants depends only on the structure of associative algebra of A , independently of a presentation of A as an N -homogeneous algebra.

The Koszul cup product is defined in Section 3. While the definition when $N = 2$ is mimicking the usual cup product, the formula giving $f \smile_K g$ is very specific – not at all obvious – when $N > 2$ and the homogeneous Koszul cochains f and g are both of odd degree. It turns out that this new cup product \smile_K is compatible with the Koszul differential b_K , allowing us to define \smile_K on Koszul cohomology classes.

A key step in the construction of the Koszul calculus for $N > 2$ is the proof of the

associativity of $\underset{K}{\smile}$ on Koszul cohomology classes (Subsection 3.2). This proof is not obvious and is rather long and technical, but it is essential in order to obtain a graded associative algebra $HK^\bullet(A)$, the space $HK^\bullet(A, M)$ becoming a graded $HK^\bullet(A)$ -bimodule for any A -bimodule M . In contrast to the Hochschild calculus [20], we do not know whether this algebra is graded commutative or this bimodule is graded symmetric.

If we are interested in the structure at cochain level, the point is that the Koszul cup product is not associative for any cochains in general (Subsection 3.3). So we arrive to a well-known situation in algebraic topology, for which associativity holds on classes and does not hold on cochains: Koszul cochains with coefficients in A should constitute an A_∞ -algebra. The details of our proof of associativity on classes should be a guide to find the ternary operation m_3 of this A_∞ -algebra. Moreover, the algebra $W_{\nu(\bullet)}^*$ of Koszul cochains with coefficients in k should be an A_∞ -algebra as well. If A is N -Koszul and finitely generated, the algebra $W_{\nu(\bullet)}^*$ is isomorphic to the Yoneda algebra $E(A)$ of A (Proposition 3.7). Therefore, it is expected that the A_∞ -algebra coming from Koszul calculus should coincide with the A_∞ -algebra defined on $E(A)$ by He and Lu [24]. We have pursued no further in this direction.

The Koszul cup bracket is defined and studied in Section 4. Like in the quadratic situation, there is no Gerstenhaber bracket in the picture. Precisely, the fundamental formula (1.1) is nicely generalized to any Koszul p -cochain f with coefficients in any A -bimodule M , as follows

1. $[e_A, f]_{\underset{K}{\smile}} = -b_K(f)$ if p is even,
2. $[e_A, f]_{\underset{K}{\smile}} = (1 - N)b_K(f)$ if p is odd.

Other formulas concerning on Koszul derivations or higher Koszul cohomology are generalized in the same manner.

The Koszul cap products are defined in Section 5. Here again, this definition is very specific if $N > 2$. This definition passes to classes, obtaining a graded $(HK^\bullet(A), \underset{K}{\smile})$ -bimodule $HK_\bullet(A, M)$ for the actions $\underset{K}{\frown}$. The proof of associativity formulas is long and technical (Subsection 5.2). The non-associativity at the chain-cochain level suggests that Koszul chains should form an A_∞ -bimodule over the A_∞ -algebra of Koszul cochains.

Koszul cap bracket, cap-actions of Koszul derivations and higher Koszul homology are treated in Section 6. The above N -generalization of Formula (1.1) has an analogue for the cap bracket. The paper ends with the study of the truncated polynomial algebra in one variable (Section 7). Although this algebra is elementary, it is instructive to calculate its Koszul calculus, with respect to some natural general questions concerning on the N -case, questions already discussed in [9] if $N = 2$.

2 Koszul homology and cohomology

Throughout the article, let us fix a vector space V over the field k and an integer $N \geq 2$. The tensor algebra $T(V) = \bigoplus_{m \geq 0} V^{\otimes m}$ is graded by the *weight* m . For any subspace R of $V^{\otimes N}$, the associative algebra $A = T(V)/(R)$ inherits the weight grading. The homogeneous component of weight m of A is denoted by A_m . The graded algebra A is called an *N -homogeneous algebra*. If $N = 2$, A is a quadratic algebra [9].

2.1 Bimodule complex $K(A)$

Let $A = T(V)/(R)$ be an N -homogeneous algebra. For any $p \geq 0$, W_p denotes the subspace of $V^{\otimes p}$ defined by

$$W_p = \bigcap_{i+N+j=p} V^{\otimes i} \otimes R \otimes V^{\otimes j}.$$

Remark that $W_p = V^{\otimes p}$ if $0 \leq p < N$, and $W_N = R$. As for quadratic algebras, an arbitrary element of W_p is denoted by a product $x_1 \dots x_p$, where x_1, \dots, x_p are in V . If $q + r + s = p$, regarding W_p as a subspace of $V^{\otimes q} \otimes W_r \otimes V^{\otimes s}$, the element $x_1 \dots x_p$ viewed in $V^{\otimes q} \otimes W_r \otimes V^{\otimes s}$ will be denoted by the *same* notation, meaning that the product $x_{q+1} \dots x_{q+r}$ is thought of as an element of W_r and the other x 's are thought of as arbitrary in V .

Define the map $\nu : \mathbb{N} \rightarrow \mathbb{N}$ by

$$\begin{aligned} \nu(p) &= Np' \text{ if } p \text{ even, } p = 2p', \\ \nu(p) &= Np' + 1 \text{ if } p \text{ odd, } p = 2p' + 1. \end{aligned}$$

The integers p and q are said to be ν -additive if $\nu(p+q) = \nu(p) + \nu(q)$. If $N > 2$, it is the case if and only if p and q are not both odd. One has

$$\begin{aligned} \nu(p) &= \nu(p-1) + 1 \text{ if } p \text{ odd,} \\ \nu(p) &= \nu(p-1) + N - 1 \text{ if } p \text{ even,} \end{aligned}$$

implying the inclusions

$$\begin{aligned} W_{\nu(p)} &\subseteq (V \otimes W_{\nu(p-1)}) \cap (W_{\nu(p-1)} \otimes V) \text{ if } p \text{ odd,} \\ W_{\nu(p)} &\subseteq \bigcap_{i+j=N-1} V^{\otimes i} \otimes W_{\nu(p-1)} \cap V^{\otimes j} \text{ if } p \text{ even.} \end{aligned}$$

The bimodule complex $K(A)$, as defined in [10], is the following

$$\dots \xrightarrow{d} K_p \xrightarrow{d} K_{p-1} \xrightarrow{d} \dots \xrightarrow{d} K_1 \xrightarrow{d} K_0 \longrightarrow 0 \quad (2.1)$$

where $K(A)_p = K_p$ denotes the space $A \otimes W_{\nu(p)} \otimes A$. For any a, a' in A and $x_1 \dots x_{\nu(p)}$ in $W_{\nu(p)}$, the differential d is defined on K_p by

$$d(a \otimes x_1 \dots x_{Np'+1} \otimes a') = ax_1 \otimes x_2 \dots x_{Np'+1} \otimes a' - a \otimes x_1 \dots x_{Np'} \otimes x_{Np'+1} a' \quad (2.2)$$

if $p = 2p' + 1$, and by

$$d(a \otimes x_1 \dots x_{Np'} \otimes a') = \sum_{0 \leq i \leq N-1} ax_1 \dots x_i \otimes x_{i+1} \dots x_{i+Np'-N+1} \otimes x_{i+Np'-N+2} \dots x_{Np'} a' \quad (2.3)$$

if $p = 2p'$. Then $K(A)$ is a weight graded complex of free A -bimodules.

Since the complex

$$A \otimes R \otimes A \xrightarrow{d} A \otimes V \otimes A \xrightarrow{d} A \otimes A \xrightarrow{\mu} A \rightarrow 0 \quad (2.4)$$

ending by the multiplication μ of A is exact, the homology of $K(A)$ is equal to A in degree 0, and to 0 in degree 1. Koszul algebras for $N > 2$ were defined in [4], and the following equivalent definition appeared in [10].

Definition 2.1 An N -homogeneous algebra $A = T(V)/(R)$ is said to be Koszul if the homology of $K(A)$ is 0 in any positive degree. A Koszul N -homogeneous algebra is also called an N -Koszul algebra.

If $R = 0$ or $R = V^{\otimes N}$, A is Koszul [4]. Besides these extreme examples, many various N -Koszul algebras are available in the literature, as the example of Subsection 3.3.

2.2 Koszul homology and cohomology of A

Let M be an A -bimodule, considered as a left or right A^e -module, where $A^e = A \otimes A^{op}$. Applying the functors $M \otimes_{A^e} -$ and $\text{Hom}_{A^e}(-, M)$ to the complex $K(A)$, we obtain the chain and cochain complexes $(M \otimes W_{\nu(\bullet)}, b_K)$ and $(\text{Hom}(W_{\nu(\bullet)}, M), b_K)$. The elements of $M \otimes W_{\nu(p)}$ and $\text{Hom}(W_{\nu(p)}, M)$ are called *Koszul p -chains* and *p -cochains with coefficients in M* , respectively. Equations (2.2) and (2.3) show that, for any p -chain $z = m \otimes x_1 \dots x_{\nu(p)}$, one has

$$b_K(z) = mx_1 \otimes x_2 \dots x_{Np'+1} - x_{Np'+1}m \otimes x_1 \dots x_{Np'} \quad (2.5)$$

if $p = 2p' + 1$, and

$$b_K(z) = \sum_{0 \leq i \leq N-1} x_{i+Np'-N+2} \dots x_{Np'} mx_1 \dots x_i \otimes x_{i+1} \dots x_{i+Np'-N+1} \quad (2.6)$$

if $p = 2p'$. Similarly, for any p -cochain f , one has

$$b_K(f)(x_1 \dots x_{Np'+1}) = f(x_1 \dots x_{Np'})x_{Np'+1} - x_1 f(x_2 \dots x_{Np'+1}) \quad (2.7)$$

if $p = 2p'$, and

$$b_K(f)(x_1 \dots x_{Np'+N}) = \sum_{0 \leq i \leq N-1} x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'+1}) x_{i+Np'+2} \dots x_{Np'+N} \quad (2.8)$$

if $p = 2p' + 1$. If $N = 2$, we recover formulas of the quadratic case [9].

Definition 2.2 Let $A = T(V)/(R)$ be an N -homogeneous algebra and M be an A -bimodule. The homology of $(M \otimes W_{\nu(\bullet)}, b_K)$, resp. $(\text{Hom}(W_{\nu(\bullet)}, M), b_K)$, is called *Koszul homology*, resp. *Koszul cohomology*, of A with coefficients in M , and is denoted by $HK_{\bullet}(A, M)$, resp. $HK^{\bullet}(A, M)$. We set $HK_{\bullet}(A) = HK_{\bullet}(A, A)$ and $HK^{\bullet}(A) = HK^{\bullet}(A, A)$.

If A is Koszul, then $HK_{\bullet}(A, M)$, resp. $HK^{\bullet}(A, M)$, is isomorphic to the Hochschild homology $HH_{\bullet}(A, M)$, resp. cohomology $HH^{\bullet}(A, M)$. Since $K(A)$ is a complex of free A -bimodules, $M \mapsto HK_{\bullet}(A, M)$ and $M \mapsto HK^{\bullet}(A, M)$ define δ -functors from the category of A -bimodules to the category of vector spaces, that is, a short exact sequence of bimodules gives rise to a long exact sequence in Koszul homology and in Koszul cohomology [33].

For any A -bimodule M , the left derived functor $M \overset{L}{\otimes}_{A^e} -$ allows us to show, as in the quadratic case with the *same* proof [9], that the Koszul homology is isomorphic to the following Hochschild hyperhomology

$$HK_{\bullet}(A, M) \cong \mathbb{H}\mathbb{H}_{\bullet}(A, M \otimes_A K(A)). \quad (2.9)$$

Similarly, the Koszul cohomology is isomorphic to the following Hochschild hypercohomology

$$HK^{\bullet}(A, M) \cong \mathbb{H}\mathbb{H}^{\bullet}(A, \text{Hom}_A(K(A), M)). \quad (2.10)$$

2.3 Small homological degrees

If $N = 2$, $K(A)$ is naturally a subcomplex of the bar resolution $B(A)$. If $N > 2$, there is a noncanonical injective morphism of complexes from $K(A)$ to $B(A)$. This morphism depends on a choice of an arbitrary contracting homotopy of the bar resolution. We just explain the beginning of the construction of such a morphism, denoted by χ . The whole construction will be performed in Section 7 for truncated polynomial algebras.

Let $A = T(V)/(R)$ be an N -homogeneous algebra. Denote by b' the differential of the normalized $\bar{B}(A)$ and by s its contracting homotopy given by the extra degeneracy [26]. In particular, $s_0 : A \otimes A \rightarrow A \otimes \bar{A} \otimes A$ and $s_1 : A \otimes \bar{A} \otimes A \rightarrow A \otimes \bar{A}^{\otimes 2} \otimes A$ are defined by

$$s_0(a \otimes a') = 1 \otimes \bar{a} \otimes a',$$

$$s_1(a \otimes a' \otimes a'') = 1 \otimes \bar{a} \otimes a' \otimes a'',$$

where $\bar{A} = A/k$ and \bar{a} is the class of $a \in A$ in \bar{A} . In what follows, the space \bar{A} is always identified to the subspace $A_+ = \bigoplus_{m>0} A_m$ of A , so that $\bar{B}(A)$ is considered as a subcomplex of $B(A)$. If $N = 2$, $K(A)$ is a subcomplex of $\bar{B}(A)$.

First, $\chi_1 : A \otimes V \otimes A \rightarrow A \otimes \bar{A} \otimes A$ is defined by the inclusion of V into \bar{A} . The A -bimodule map $\chi_2 : A \otimes R \otimes A \rightarrow A \otimes \bar{A}^{\otimes 2} \otimes A$ is defined by its restriction to R , which is equal to $s_1 \circ \chi_1 \circ d$. Then the following diagram commutes

$$\begin{array}{ccccc} A \otimes R \otimes A & \xrightarrow{d} & A \otimes V \otimes A & \xrightarrow{d} & A \otimes A \longrightarrow 0 \\ \chi_2 \downarrow & & \chi_1 \downarrow & & id \downarrow \\ A \otimes \bar{A}^{\otimes 2} \otimes A & \xrightarrow{b'} & A \otimes \bar{A} \otimes A & \xrightarrow{b'} & A \otimes A \longrightarrow 0. \end{array} \quad (2.11)$$

Applying the functors $M \otimes_{A^e} -$ and $Hom_{A^e}(-, M)$ to this diagram, we obtain the following ones

$$\begin{array}{ccccc} M \otimes R & \xrightarrow{b_K} & M \otimes V & \xrightarrow{b_K} & M \longrightarrow 0 \\ \tilde{\chi}_2 \downarrow & & \tilde{\chi}_1 \downarrow & & id \downarrow \\ M \otimes \bar{A}^{\otimes 2} & \xrightarrow{b} & M \otimes \bar{A} & \xrightarrow{b} & M \longrightarrow 0, \end{array} \quad (2.12)$$

$$\begin{array}{ccccc} 0 \longrightarrow & M & \xrightarrow{b} & Hom(\bar{A}, M) & \xrightarrow{b} Hom(\bar{A}^{\otimes 2}, M) \\ & id \downarrow & & \chi_1^* \downarrow & \chi_2^* \downarrow \\ 0 \longrightarrow & M & \xrightarrow{b_K} & Hom(V, M) & \xrightarrow{b_K} Hom(R, M), \end{array} \quad (2.13)$$

where $\tilde{\chi}_1$, resp. χ_1^* , is the natural injection, resp. projection. From

$$\chi_2(a \otimes x_1 \dots x_N \otimes a') = \sum_{1 \leq i \leq N-1} a \otimes x_1 \dots x_i \otimes x_{i+1} \otimes x_{i+2} \dots x_N a'$$

for any a, a' in A and $x_1 \dots x_N$ in R , we draw the expressions

$$\tilde{\chi}_2(m \otimes x_1 \dots x_N) = \sum_{1 \leq i \leq N-1} x_{i+2} \dots x_N m \otimes (x_1 \dots x_i \otimes x_{i+1}),$$

$$\chi_2^*(f)(x_1 \dots x_N) = \sum_{1 \leq i \leq N-1} f(x_1 \dots x_i \otimes x_{i+1}) x_{i+2} \dots x_N.$$

Since the two lines of (2.11) are beginning projective bimodule resolutions of A , $H(\tilde{\chi}_p) : HK_p(A, M) \rightarrow HH_p(A, M)$ and $H(\chi_p^*) : HH^p(A, M) \rightarrow HK^p(A, M)$ are isomorphisms for $p = 0$ and $p = 1$.

2.4 Functoriality

Denote by \mathcal{C} the generalized Manin category of N -homogeneous algebras – introduced in [7] – and by \mathcal{E} the category of graded vector spaces. Recall that in \mathcal{C} , the objects are the N -homogeneous algebras and the morphisms are the morphisms of graded algebras. The A -bimodule $A^* = \text{Hom}(A, k)$ is defined by the actions $(a.f.a')(x) = f(a'xa)$ for any linear map $f : A \rightarrow k$, and x, a, a' in A . In the following statement, A^* may be replaced by the graded dual of A , using graded Hom in the proof.

Proposition 2.3 *The maps $A \mapsto HK_\bullet(A)$ and $A \mapsto HK^\bullet(A, A^*)$ define functors from \mathcal{C} to \mathcal{E} .*

Proof. Let $A = T(V)/(R)$ and $A' = T(V')/(R')$ be N -homogeneous algebras. A morphism u from A to A' in \mathcal{C} is entirely defined by a linear map $u_1 : V \rightarrow V'$ such that $u_1^{\otimes N}(R) \subseteq R'$. For each p , $u_1^{\otimes p}$ maps W_p into W'_p , with obvious notation. Then the maps $a \otimes x_1 \dots x_{\nu(p)} \mapsto u(a) \otimes u(x_1) \dots u(x_{\nu(p)})$ define the morphism of complexes u_\bullet from $(A \otimes W_{\nu(\bullet)}, b_K)$ to $(A' \otimes W'_{\nu(\bullet)}, b_K)$. So we obtain a covariant functor $A \mapsto HK_\bullet(A)$.

For each p , one has the linear isomorphism

$$\eta_p : \text{Hom}(A \otimes W_{\nu(p)}, k) \rightarrow \text{Hom}(W_{\nu(p)}, A^*)$$

defined by $\eta_p(f)(x_1 \dots x_{\nu(p)})(a) = f(a \otimes x_1 \dots x_{\nu(p)})$. Recall that the differential b_K^* of the complex $\text{Hom}(A \otimes W_{\nu(\bullet)}, k)$, dual to the complex $(A \otimes W_{\nu(\bullet)}, b_K)$, is defined by $b_K^*(f) = -(-1)^p f \circ b_K$ for any f in $\text{Hom}(A \otimes W_{\nu(p)}, k)$. Then it is immediate to verify that

$$\eta : (\text{Hom}(A \otimes W_{\nu(\bullet)}, k), b_K^*) \rightarrow (\text{Hom}(W_{\nu(\bullet)}, A^*), b_K)$$

is an isomorphism of complexes. Via this isomorphism, the dual of the morphism u_\bullet gives rise to the morphism of complexes u^\bullet from $(\text{Hom}(W'_{\nu(\bullet)}, A'^*), b_K)$ to $(\text{Hom}(W_{\nu(\bullet)}, A^*), b_K)$. So we obtain a contravariant functor $A \mapsto HK^\bullet(A, A^*)$. ■

Let $A = T(V)/(R)$ and $A' = T(V')/(R')$ be N -homogeneous algebras. An isomorphism u from A to A' in \mathcal{C} is entirely defined by a linear isomorphism $u_1 : V \rightarrow V'$ such that $u_1^{\otimes N}(R) = R'$. As in the previous proof, for any A -bimodule M , u defines naturally a complex isomorphism from $(M \otimes W_{\nu(\bullet)}, b_K)$ to $(M \otimes W'_{\nu(\bullet)}, b_K)$, where M is an A' -bimodule via u , inducing natural isomorphisms $HK_\bullet(A, M) \cong HK_\bullet(A', M)$. Similarly, u induces natural isomorphisms $HK^\bullet(A', M) \cong HK^\bullet(A, M)$. It is clear from Definition 3.1 and Definition 5.1 that these isomorphisms respect the Koszul cup and cap products. To resume all these properties, we say that a Manin isomorphism induces isomorphic Koszul calculus, or that *the Koszul calculus is an invariant of Manin's category*.

2.5 A more general invariance

It is based on the following result recently obtained by Bell and Zhang [3].

Theorem 2.4 *Let B and B' be two connected graded algebras over a field, finitely generated in degree 1. If $B \cong B'$ as ungraded algebras, then $B \cong B'$ as graded algebras.*

Let A be an associative algebra having a finite N -homogeneous presentation B , i.e., A is isomorphic to an N -homogeneous algebra $B = T(V)/(R)$ with V finite-dimensional. Then we can define the Koszul calculus of A as being the Koszul calculus of B . In fact, Theorem 2.4 and Manin's invariance show that the so-defined Koszul calculus of A does not depend on the choice of a finite N -homogeneous presentation of A .

2.6 Coefficients in k

We describe briefly how the same results extend from the quadratic case [9] to the N case. As in the quadratic case, the differentials b_K vanish if $M = k$. We denote by E^* the dual vector space of a vector space E .

Proposition 2.5 *Let $A = T(V)/(R)$ be an N -homogeneous algebra. For any $p \geq 0$, one has $HK_p(A, k) = W_{\nu(p)}$ and $HK^p(A, k) = W_{\nu(p)}^*$.*

In the category of graded A -bimodules, A has a minimal projective resolution $P(A)$ whose component of homological degree p has the form $A \otimes E_p \otimes A$, where E_p is a weight graded space. Moreover, the minimal weight in E_p is equal to $\nu(p)$ and the component of weight $\nu(p)$ in E_p contains $W_{\nu(p)}$ [4, 23]. So $K(A)$ is naturally a weight graded subcomplex of $P(A)$.

Denote by \underline{Hom} the graded Hom w.r.t. the weight grading of A , and by \underline{HH} the corresponding graded Hochschild cohomology HH . A fundamental property of the minimality of $P(A)$ is that the differentials of the complexes $k \otimes_{A^e} P(A)$ and $\underline{Hom}_{A^e}(P(A), k)$ vanish. Consequently, we have $HH_p(A, k) \cong E_p$ and $\underline{HH}^p(A, k) \cong \underline{Hom}(E_p, k)$ for any $p \geq 0$.

Therefore, denoting by ι the inclusion of $K(A)$ into $P(A)$, $H(\tilde{\iota})_p$ coincides with the natural injection of W_p into E_p and $H(\iota^*)_p$ with the natural projection of $\underline{Hom}(E_p, k)$ onto W_p^* . We thus obtain the following characterizations.

Proposition 2.6 *Let $A = T(V)/(R)$ be an N -homogeneous algebra. The algebra A is Koszul if and only if one of the following properties holds.*

- (i) *For any $p \geq 0$, $H(\tilde{\iota})_p : HK_p(A, k) \rightarrow HH_p(A, k)$ is an isomorphism.*
- (ii) *For any $p \geq 0$, $H(\iota^*)_p : \underline{HH}^p(A, k) \rightarrow HK^p(A, k)$ is an isomorphism.*

3 Koszul cup product

3.1 Definition and first properties

Definition 3.1 *Let $A = T(V)/(R)$ be an N -homogeneous algebra. Let P and Q be A -bimodules. For Koszul cochains $f : W_{\nu(p)} \rightarrow P$ and $g : W_{\nu(q)} \rightarrow Q$, we define the Koszul*

$(p+q)$ -cochain $f \underset{K}{\smile} g : W_{\nu(p+q)} \rightarrow P \otimes_A Q$ by

1. if p and q are not both odd, so that $\nu(p+q) = \nu(p) + \nu(q)$, one has

$$(f \underset{K}{\smile} g)(x_1 \dots x_{\nu(p+q)}) = f(x_1 \dots x_{\nu(p)}) \otimes_A g(x_{\nu(p)+1} \dots x_{\nu(p)+\nu(q)}),$$

2. if p and q are both odd, so that $\nu(p+q) = \nu(p) + \nu(q) + N - 2$, one has

$$(f \underset{K}{\smile} g)(x_1 \dots x_{\nu(p+q)}) = - \sum_{0 \leq i+j \leq N-2} x_1 \dots x_i f(x_{i+1} \dots x_{i+\nu(p)}) x_{i+\nu(p)+1} \dots x_{\nu(p)+N-j-2} \\ \otimes_A g(x_{\nu(p)+N-j-1} \dots x_{\nu(p)+\nu(q)+N-j-2}) x_{\nu(p)+\nu(q)+N-j-1} \dots x_{\nu(p)+\nu(q)+N-2}.$$

The k -bilinear product $\underset{K}{\smile}$ is called N -Koszul cup product, or simply *Koszul cup product* if N is clearly specified. When $N = 2$, it coincides with the Koszul cup product defined in [9]. When $N > 2$, if p and q are not both odd, $f \underset{K}{\smile} g$ coincides up to a sign with the restriction of the usual cup product $f \smile g$ to $W_{\nu(p+q)}$, but if p and q are odd, $f \underset{K}{\smile} g$ is new.

When $N = 2$, $\underset{K}{\smile}$ is associative. When $N > 2$, the N -Koszul cup product may be *non-associative* (Subsection 3.3). However, we will prove in Subsection 3.2 that it is associative on Koszul cohomology classes. Our first task is to prove that $\underset{K}{\smile}$ passes to classes.

Proposition 3.2 *Let $A = T(V)/(R)$ be an N -homogeneous algebra. Let P and Q be A -bimodules. For any Koszul p -cochain f with coefficients in P and q -cochain g with coefficients in Q , one has*

$$b_K(f \underset{K}{\smile} g) = b_K(f) \underset{K}{\smile} g + (-1)^p f \underset{K}{\smile} b_K(g). \quad (3.1)$$

Proof. 1. Assume $p = 2p'$ and $q = 2q'$. From (2.7) and Definition 3.1, we get

$$b_K(f \underset{K}{\smile} g)(x_1 \dots x_{Np'+Nq'+1}) = f(x_1 \dots x_{Np'}) \otimes_A g(x_{Np'+1} \dots x_{Np'+Nq'}) x_{Np'+Nq'+1} \\ - x_1 f(x_2 \dots x_{Np'+1}) \otimes_A g(x_{Np'+2} \dots x_{Np'+Nq'+1}), \\ b_K(f) \underset{K}{\smile} g(x_1 \dots x_{Np'+Nq'+1}) = f(x_1 \dots x_{Np'}) x_{Np'+1} \otimes_A g(x_{Np'+2} \dots x_{Np'+Nq'+1}) \\ - x_1 f(x_2 \dots x_{Np'+1}) \otimes_A g(x_{Np'+2} \dots x_{Np'+Nq'+1}), \\ f \underset{K}{\smile} b_K(g)(x_1 \dots x_{Np'+Nq'+1}) = f(x_1 \dots x_{Np'}) \otimes_A g(x_{Np'+1} \dots x_{Np'+Nq'}) x_{Np'+Nq'+1} \\ - f(x_1 \dots x_{Np'}) \otimes_A x_{Np'+1} g(x_{Np'+2} \dots x_{Np'+Nq'+1}),$$

proving (3.1) in this case.

2. Assume $p = 2p'$ and $q = 2q' + 1$. From (2.7), (2.8) and Definition 3.1, we get on one hand

$$b_K(f \underset{K}{\smile} g)(x_1 \dots x_{Np'+Nq'+N}) = \sum_{0 \leq i \leq N-1} x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'}) \\ \otimes_A g(x_{i+Np'+1} \dots x_{i+Np'+Nq'+1}) x_{i+Np'+Nq'+2} \dots x_{Np'+Nq'+N}. \quad (3.2)$$

On the other hand, let us calculate firstly

$$\begin{aligned}
& b_K(f) \underset{K}{\smile} g(x_1 \dots x_{Np'+Nq'+N}) \\
&= \sum_{0 \leq i+j \leq N-2} -x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'}) x_{i+Np'+1} \dots x_{Np'+N-j-1} \\
&\quad \otimes_A g(x_{Np'+N-j} \dots x_{Np'+Nq'+N-j}) x_{Np'+Nq'+N-j+1} \dots x_{Np'+Nq'+N} \\
&\quad + x_1 \dots x_{i+1} f(x_{i+2} \dots x_{i+Np'+1}) x_{i+Np'+2} \dots x_{Np'+N-j-1} \\
&\quad \otimes_A g(x_{Np'+N-j} \dots x_{Np'+Nq'+N-j}) x_{Np'+Nq'+N-j+1} \dots x_{Np'+Nq'+N}.
\end{aligned}$$

Therefore, we obtain a telescopic sum which is easily reduced to

$$\begin{aligned}
& b_K(f) \underset{K}{\smile} g(x_1 \dots x_{Np'+Nq'+N}) \\
&= \sum_{0 \leq j \leq N-2} -f(x_1 \dots x_{Np'}) x_{Np'+1} \dots x_{Np'+N-j-1} \\
&\quad \otimes_A g(x_{Np'+N-j} \dots x_{Np'+Nq'+N-j}) x_{Np'+Nq'+N-j+1} \dots x_{Np'+Nq'+N} \\
&\quad + \sum_{0 \leq j \leq N-2} x_1 \dots x_{N-j-1} f(x_{N-j} \dots x_{Np'+N-j-1}) \\
&\quad \otimes_A g(x_{Np'+N-j} \dots x_{Np'+Nq'+N-j}) x_{Np'+Nq'+N-j+1} \dots x_{Np'+Nq'+N}.
\end{aligned}$$

Secondly, the right-hand side of

$$\begin{aligned}
& f \underset{K}{\smile} b_K(g)(x_1 \dots x_{Np'+Nq'+N}) = \sum_{0 \leq i \leq N-1} f(x_1 \dots x_{Np'}) \\
&\quad \otimes_A x_{Np'+1} \dots x_{Np'+i} g(x_{Np'+i+1} \dots x_{Np'+Nq'+i+1}) x_{Np'+Nq'+i+2} \dots x_{Np'+Nq'+N}
\end{aligned}$$

is rewritten as

$$\begin{aligned}
& f \underset{K}{\smile} b_K(g)(x_1 \dots x_{Np'+Nq'+N}) \\
&= f(x_1 \dots x_{Np'}) \otimes_A g(x_{Np'+1} \dots x_{Np'+Nq'+1}) x_{Np'+Nq'+2} \dots x_{Np'+Nq'+N} \\
&\quad + \sum_{0 \leq j \leq N-2} f(x_1 \dots x_{Np'}) x_{Np'+1} \dots x_{Np'+N-j-1} \\
&\quad \otimes_A g(x_{Np'+N-j} \dots x_{Np'+Nq'+N-j}) x_{Np'+Nq'+N-j+1} \dots x_{Np'+Nq'+N}.
\end{aligned}$$

Putting together the so-obtained formulas, we arrive to

$$\begin{aligned}
& (b_K(f) \underset{K}{\smile} g + f \underset{K}{\smile} b_K(g))(x_1 \dots x_{Np'+Nq'+N}) \\
&= \sum_{0 \leq j \leq N-1} x_1 \dots x_{N-j-1} f(x_{N-j} \dots x_{Np'+N-j-1}) \\
&\quad \otimes_A g(x_{Np'+N-j} \dots x_{Np'+Nq'+N-j}) x_{Np'+Nq'+N-j+1} \dots x_{Np'+Nq'+N}
\end{aligned}$$

which is equal to the right-hand side of (3.2) by setting $i = N - 1 - j$.

3. Assume $p = 2p' + 1$ and $q = 2q'$. On one hand, one has

$$\begin{aligned} b_K(f \underset{K}{\smile} g)(x_1 \dots x_{Np'+Nq'+N}) &= \sum_{0 \leq i \leq N-1} x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'+1}) \\ &\quad \otimes_A g(x_{i+Np'+2} \dots x_{i+Np'+Nq'+1}) x_{i+Np'+Nq'+2} \dots x_{Np'+Nq'+N}. \end{aligned} \quad (3.3)$$

On the other hand, let us write firstly

$$\begin{aligned} b_K(f) \underset{K}{\smile} g(x_1 \dots x_{Np'+Nq'+N}) &= \sum_{0 \leq i \leq N-1} x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'+1}) \\ &\quad \otimes_A x_{i+Np'+2} \dots x_{Np'+N} g(x_{Np'+N+1} \dots x_{Np'+Nq'+N}), \end{aligned}$$

and secondly

$$\begin{aligned} f \underset{K}{\smile} b_K(g)(x_1 \dots x_{Np'+Nq'+N}) &= \sum_{0 \leq i+j \leq N-2} -x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'+1}) x_{i+Np'+2} \dots x_{Np'+N-j-1} \\ &\quad \otimes_A g(x_{Np'+N-j} \dots x_{Np'+Nq'+N-j-1}) x_{Np'+Nq'+N-j} \dots x_{Np'+Nq'+N} \\ &\quad + x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'+1}) x_{i+Np'+2} \dots x_{Np'+N-j} \\ &\quad \otimes_A g(x_{Np'+N-j+1} \dots x_{Np'+Nq'+N-j}) x_{Np'+Nq'+N-j+1} \dots x_{Np'+Nq'+N} \end{aligned}$$

reduced telescopically to

$$\begin{aligned} f \underset{K}{\smile} b_K(g)(x_1 \dots x_{Np'+Nq'+N}) &= \sum_{0 \leq i \leq N-2} -x_1 \dots x_i f(x_{i+1} \dots x_{Np'+i+1}) \\ &\quad \otimes_A g(x_{Np'+i+2} \dots x_{Np'+Nq'+i+1}) x_{Np'+Nq'+i+2} \dots x_{Np'+Nq'+N} \\ &\quad + \sum_{0 \leq i \leq N-2} x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'+1}) x_{i+Np'+2} \dots x_{Np'+N} \\ &\quad \otimes_A g(x_{Np'+N+1} \dots x_{Np'+Nq'+N}). \end{aligned}$$

So we obtain

$$\begin{aligned} (b_K(f) \underset{K}{\smile} g - f \underset{K}{\smile} b_K(g))(x_1 \dots x_{Np'+Nq'+N}) &= \sum_{0 \leq i \leq N-1} x_1 \dots x_i f(x_{i+1} \dots x_{Np'+i+1}) \\ &\quad \otimes_A g(x_{Np'+i+2} \dots x_{Np'+Nq'+i+1}) x_{Np'+Nq'+i+2} \dots x_{Np'+Nq'+N} \end{aligned}$$

which coincides with the right-hand side of (3.3).

4. Assume $p = 2p' + 1$ and $q = 2q' + 1$. On one hand, we have

$$\begin{aligned}
& b_K(f \underset{K}{\smile} g)(x_1 \dots x_{Np'+Nq'+N+1}) \\
&= \sum_{0 \leq i+j \leq N-2} -x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'+1}) x_{i+Np'+2} \dots x_{Np'+N-j-1} \\
&\quad \otimes_A g(x_{Np'+N-j} \dots x_{Np'+Nq'+N-j}) x_{Np'+Nq'+N-j+1} \dots x_{Np'+Nq'+N+1} \\
&\quad + x_1 \dots x_{i+1} f(x_{i+2} \dots x_{i+Np'+2}) x_{i+Np'+3} \dots x_{Np'+N-j} \\
&\quad \otimes_A g(x_{Np'+N-j+1} \dots x_{Np'+Nq'+N-j+1}) x_{Np'+Nq'+N-j+2} \dots x_{Np'+Nq'+N+1},
\end{aligned}$$

and reducing the telescopic sum, we arrive to

$$\begin{aligned}
& b_K(f \underset{K}{\smile} g)(x_1 \dots x_{Np'+Nq'+N+1}) \\
&= - \sum_{0 \leq j \leq N-2} f(x_1 \dots x_{Np'+1}) x_{Np'+2} \dots x_{Np'+N-j-1} \\
&\quad \otimes_A g(x_{Np'+N-j} \dots x_{Np'+Nq'+N-j}) x_{Np'+Nq'+N-j+1} \dots x_{Np'+Nq'+N+1} \\
&\quad + \sum_{0 \leq i \leq N-2} x_1 \dots x_{i+1} f(x_{i+2} \dots x_{i+Np'+2}) x_{i+Np'+3} \dots x_{Np'+N} \\
&\quad \otimes_A g(x_{Np'+N+1} \dots x_{Np'+Nq'+N+1}).
\end{aligned}$$

On the other hand, we have the two equalities

$$\begin{aligned}
& b_K(f) \underset{K}{\smile} g(x_1 \dots x_{Np'+Nq'+N+1}) = \sum_{0 \leq i \leq N-1} x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'+1}) \\
&\quad \otimes_A x_{i+Np'+2} \dots x_{Np'+N} g(x_{Np'+N+1} \dots x_{Np'+Nq'+N+1}),
\end{aligned}$$

$$\begin{aligned}
& f \underset{K}{\smile} b_K(g)(x_1 \dots x_{Np'+Nq'+N+1}) = \sum_{0 \leq i \leq N-1} f(x_1 \dots x_{Np'+1}) \\
&\quad \otimes_A x_{Np'+2} \dots x_{Np'+i+1} g(x_{Np'+i+2} \dots x_{Np'+Nq'+i+2}) x_{Np'+Nq'+i+3} \dots x_{Np'+Nq'+N+1}
\end{aligned}$$

which are combined as follows

$$\begin{aligned}
& (b_K(f) \underset{K}{\smile} g - f \underset{K}{\smile} b_K(g))(x_1 \dots x_{Np'+Nq'+N+1}) \\
&= \sum_{1 \leq i \leq N-1} x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'+1}) \\
&\quad \otimes_A x_{i+Np'+2} \dots x_{Np'+N} g(x_{Np'+N+1} \dots x_{Np'+Nq'+N+1}) \\
&\quad - \sum_{0 \leq i \leq N-2} f(x_1 \dots x_{Np'+1}) \\
&\quad \otimes_A x_{Np'+2} \dots x_{Np'+i+1} g(x_{Np'+i+2} \dots x_{Np'+Nq'+i+2}) x_{Np'+Nq'+i+3} \dots x_{Np'+Nq'+N+1}.
\end{aligned}$$

It suffices to replace i by $i+1$ in the first sum, and by $N-2-j$ in the second one, to recover $b_K(f \underset{K}{\smile} g)(x_1 \dots x_{Np'+Nq'+N+1})$. ■

Consequently, the Koszul cup product defines a Koszul cup product, still denoted by $\underset{K}{\smile}$, on Koszul cohomology classes. Our aim is now to prove the associativity of $\underset{K}{\smile}$ on classes in the nontrivial case $N > 2$.

3.2 Associativity on cohomology classes

Let $A = T(V)/(R)$ be an N -homogeneous algebra with $N > 2$. Let M, P and Q be A -bimodules. For Koszul cochains $f : W_{\nu(p)} \rightarrow M, g : W_{\nu(q)} \rightarrow P$ and $h : W_{\nu(r)} \rightarrow Q$, their associator is the Koszul $(p + q + r)$ -cochain with coefficients in $M \otimes_A P \otimes_A Q$ defined by

$$as(f, g, h) = (f \underset{K}{\smile} g) \underset{K}{\smile} h - f \underset{K}{\smile} (g \underset{K}{\smile} h).$$

Assume that the integers p, q and r are ν -additive, meaning that $\nu(p+q+r) = \nu(p) + \nu(q) + \nu(r)$. It is the case if and only if at most one of these integers is odd. Then $as(f, g, h) = 0$ since $\underset{K}{\smile}$ coincides up to a sign with \smile in all the concerned calculations.

It remains to examine the following four cases.

1. $p = 2p', q = 2q' + 1$ and $r = 2r' + 1$. We have the two equalities

$$\begin{aligned} (f \underset{K}{\smile} g) \underset{K}{\smile} h(x_1 \dots x_{Np'+Nq'+Nr'+N}) &= - \sum_{0 \leq i+j \leq N-2} x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'}) \\ &\quad \otimes_A g(x_{i+Np'+1} \dots x_{i+Np'+Nq'+1}) x_{i+Np'+Nq'+2} \dots x_{Np'+Nq'+N-j-1} \\ &\quad \otimes_A h(x_{Np'+Nq'+N-j} \dots x_{Np'+Nq'+Nr'+N-j}) x_{Np'+Nq'+Nr'+N-j+1} \dots x_{Np'+Nq'+Nr'+N}, \\ f \underset{K}{\smile} (g \underset{K}{\smile} h)(x_1 \dots x_{Np'+Nq'+Nr'+N}) &= - \sum_{0 \leq i+j \leq N-2} f(x_1 \dots x_{Np'}) \\ &\quad \otimes_A x_{Np'+1} \dots x_{Np'+i} g(x_{Np'+i+1} \dots x_{Np'+Nq'+i+1}) x_{Np'+Nq'+i+2} \dots x_{Np'+Nq'+N-j-1} \\ &\quad \otimes_A h(x_{Np'+Nq'+N-j} \dots x_{Np'+Nq'+Nr'+N-j}) x_{Np'+Nq'+Nr'+N-j+1} \dots x_{Np'+Nq'+Nr'+N}. \end{aligned}$$

Let us write

$$as(f, g, h)(x_1 \dots x_{Np'+Nq'+Nr'+N}) = \sum_{0 \leq i \leq N-2} X_i \otimes_A Y_i,$$

$$X_i = f(x_1 \dots x_{Np'}) x_{Np'+1} \dots x_{Np'+i} - x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'}),$$

$$\begin{aligned} Y_i &= \sum_{0 \leq j \leq N-2-i} g(x_{i+Np'+1} \dots x_{i+Np'+Nq'+1}) x_{i+Np'+Nq'+2} \dots x_{Np'+Nq'+N-j-1} \\ &\quad \otimes_A h(x_{Np'+Nq'+N-j} \dots x_{Np'+Nq'+Nr'+N-j}) x_{Np'+Nq'+Nr'+N-j+1} \dots x_{Np'+Nq'+Nr'+N}. \end{aligned}$$

Clearly, $X_0 = 0, X_1 = b_K(f)(x_1 \dots x_{Np'+1})$, and more generally

$$X_i = \sum_{1 \leq \ell \leq i} x_1 \dots x_{\ell-1} b_K(f)(x_\ell \dots x_{Np'+\ell}) x_{Np'+\ell+1} \dots x_{Np'+i},$$

so that we obtain the following.

Lemma 3.3 *$as(f, g, h)(x_1 \dots x_{Np'+Nq'+Nr'+N}) = 0$ whenever f is a Koszul cocycle.*

2. $p = 2p' + 1, q = 2q'$ and $r = 2r' + 1$. We have the two equalities

$$\begin{aligned} (f \underset{K}{\smile} g) \underset{K}{\smile} h(x_1 \dots x_{Np'+Nq'+Nr'+N}) &= - \sum_{0 \leq i+j \leq N-2} x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'+1}) \\ &\quad \otimes_A g(x_{i+Np'+2} \dots x_{i+Np'+Nq'+1}) x_{i+Np'+Nq'+2} \dots x_{Np'+Nq'+N-j-1} \\ &\quad \otimes_A h(x_{Np'+Nq'+N-j} \dots x_{Np'+Nq'+Nr'+N-j}) x_{Np'+Nq'+Nr'+N-j+1} \dots x_{Np'+Nq'+Nr'+N}, \end{aligned}$$

$$\begin{aligned}
f \underset{K}{\smile} (g \underset{K}{\smile} h)(x_1 \dots x_{Np'+Nq'+Nr'+N}) &= - \sum_{0 \leq i+j \leq N-2} x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'+1}) \\
&\quad \otimes_A x_{i+Np'+2} \dots x_{Np'+N-j-1} g(x_{Np'+N-j} \dots x_{Np'+Nq'+N-j-1}) \\
&\quad \otimes_A h(x_{Np'+Nq'+N-j} \dots x_{Np'+Nq'+Nr'+N-j}) x_{Np'+Nq'+Nr'+N-j+1} \dots x_{Np'+Nq'+Nr'+N}.
\end{aligned}$$

Therefore, we may write

$$\begin{aligned}
as(f, g, h)(x_1 \dots x_{Np'+Nq'+Nr'+N}) &= \sum_{0 \leq i+j \leq N-2} Y_i \otimes_A X_{ij} \otimes_A Z_j, \\
Y_i &= x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'+1}), \\
X_{ij} &= x_{i+Np'+2} \dots x_{Np'+N-j-1} g(x_{Np'+N-j} \dots x_{Np'+Nq'+N-j-1}) \\
&\quad - g(x_{i+Np'+2} \dots x_{i+Np'+Nq'+1}) x_{i+Np'+Nq'+2} \dots x_{Np'+Nq'+N-j-1}, \\
Z_j &= h(x_{Np'+Nq'+N-j} \dots x_{Np'+Nq'+Nr'+N-j}) x_{Np'+Nq'+Nr'+N-j+1} \dots x_{Np'+Nq'+Nr'+N}.
\end{aligned}$$

Then the formula

$$\begin{aligned}
X_{ij} &= - \sum_{0 \leq \ell \leq N-2-i-j} x_{i+Np'+2} \dots x_{i+Np'+\ell} \\
&\quad b_K(g)(x_{i+Np'+\ell+1} \dots x_{i+Np'+Nq'+\ell+1}) x_{i+Np'+Nq'+\ell+2} \dots x_{Np'+Nq'+N-j-1},
\end{aligned}$$

shows the following

Lemma 3.4 $as(f, g, h)(x_1 \dots x_{Np'+Nq'+Nr'+N}) = 0$ whenever g is a Koszul cocycle.

3. $p = 2p' + 1$, $q = 2q' + 1$ and $r = 2r'$. From the two equalities

$$\begin{aligned}
(f \underset{K}{\smile} g) \underset{K}{\smile} h(x_1 \dots x_{Np'+Nq'+Nr'+N}) &= - \sum_{0 \leq i+j \leq N-2} x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'+1}) \\
&\quad \otimes_A x_{i+Np'+2} \dots x_{Np'+N-j-1} g(x_{Np'+N-j} \dots x_{Np'+Nq'+N-j}) \\
&\quad \otimes_A x_{Np'+Nq'+N-j+1} \dots x_{Np'+Nq'+N} h(x_{Np'+Nq'+N+1} \dots x_{Np'+Nq'+Nr'+N}), \\
f \underset{K}{\smile} (g \underset{K}{\smile} h)(x_1 \dots x_{Np'+Nq'+Nr'+N}) &= - \sum_{0 \leq i+j \leq N-2} x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'+1}) \\
&\quad \otimes_A x_{i+Np'+2} \dots x_{Np'+N-j-1} g(x_{Np'+N-j} \dots x_{Np'+Nq'+N-j}) \\
&\quad \otimes_A h(x_{Np'+Nq'+N-j+1} \dots x_{Np'+Nq'+Nr'+N-j}) \dots x_{Np'+Nq'+Nr'+N},
\end{aligned}$$

we draw

$$as(f, g, h)(x_1 \dots x_{Np'+Nq'+Nr'+N}) = \sum_{0 \leq j \leq N-2} Y_j \otimes_A X_j,$$

$$\begin{aligned}
X_j &= h(x_{Np'+Nq'+N-j+1} \dots x_{Np'+Nq'+Nr'+N-j}) \dots x_{Np'+Nq'+Nr'+N} \\
&\quad - x_{Np'+Nq'+N-j+1} \dots x_{Np'+Nq'+N} h(x_{Np'+Nq'+N+1} \dots x_{Np'+Nq'+Nr'+N}),
\end{aligned}$$

$$Y_j = \sum_{0 \leq i \leq N-2-j} x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'+1}) \\ \otimes_A x_{i+Np'+2} \dots x_{Np'+N-j-1} g(x_{Np'+N-j} \dots x_{Np'+Nq'+N-j}).$$

Then next lemma comes from the formula

$$X_j = \sum_{1 \leq \ell \leq j} x_{Np'+Nq'+N-j+1} \dots x_{Np'+Nq'+N-j+\ell-1} \\ b_K(h)(x_{Np'+Nq'+N-j+\ell} \dots x_{Np'+Nq'+Nr'+N-j+\ell}) \dots x_{Np'+Nq'+N}.$$

Lemma 3.5 *as(f, g, h)(x_1 \dots x_{Np'+Nq'+Nr'+N}) = 0 whenever h is a Koszul cocycle.*

4. $p = 2p' + 1$, $q = 2q' + 1$ and $r = 2r' + 1$. We have

$$(f \underset{K}{\smile} g) \underset{K}{\smile} h(x_1 \dots x_{Np'+Nq'+Nr'+N+1}) = - \sum_{0 \leq i+j \leq N-2} x_1 \dots x_i f(x_{i+1} \dots x_{i+Np'+1}) \\ \otimes_A x_{i+Np'+2} \dots x_{Np'+N-j-1} g(x_{Np'+N-j} \dots x_{Np'+Nq'+N-j}) \\ \otimes_A x_{Np'+Nq'+N-j+1} \dots x_{Np'+Nq'+N} h(x_{Np'+Nq'+N+1} \dots x_{Np'+Nq'+Nr'+N+1}) \quad (3.4)$$

$$f \underset{K}{\smile} (g \underset{K}{\smile} h)(x_1 \dots x_{Np'+Nq'+Nr'+N+1}) = - \sum_{0 \leq i+j \leq N-2} f(x_1 \dots x_{Np'+1}) \\ \otimes_A x_{Np'+2} \dots x_{Np'+i+1} g(x_{Np'+i+2} \dots x_{Np'+Nq'+i+2}) x_{Np'+Nq'+i+3} \dots x_{Np'+Nq'+N-j} \\ \otimes_A h(x_{Np'+Nq'+N-j+1} \dots x_{Np'+Nq'+Nr'+N-j+1}) \dots x_{Np'+Nq'+Nr'+N+1} \quad (3.5)$$

Setting $m = Np' + Nq' + Nr' + N$, we want to show that

$$E = as(f, g, h)(x_1 \dots x_{m+1})$$

is a sum of Koszul coboundaries $b_K(u)$, where $u : W_m \rightarrow M \otimes_A P \otimes_A Q$ will be a $(p+q+r-1)$ -cochain depending on f , g and h , so that $b_K(u)$ will be defined by

$$b_K(u)(x_1 \dots x_{m+1}) = u(x_1 \dots x_m) x_{m+1} - x_1 u(x_2 \dots x_{m+1}).$$

In (3.4), i is replaced by k . Next, for k fixed, $N - 2 - k - j$ is replaced by i , obtaining

$$(f \underset{K}{\smile} g) \underset{K}{\smile} h(x_1 \dots x_{m+1}) = - \sum_{k=0}^{N-2} \sum_{i=0}^{N-2-k} x_1 \dots x_k f(x_{k+1} \dots x_{k+Np'+1}) \\ \otimes_A x_{k+Np'+2} \dots x_{Np'+i+k+1} g(x_{Np'+i+k+2} \dots x_{Np'+Nq'+i+k+2}) \\ \otimes_A x_{Np'+Nq'+i+k+3} \dots x_{Np'+Nq'+N} h(x_{Np'+Nq'+N+1} \dots x_{m+1}).$$

In (3.5), replacing j by k , we obtain

$$f \underset{K}{\smile} (g \underset{K}{\smile} h)(x_1 \dots x_{Np'+Nq'+Nr'+N+1}) = - \sum_{k=0}^{N-2} \sum_{i=0}^{N-2-k} f(x_1 \dots x_{Np'+1}) \\ \otimes_A x_{Np'+2} \dots x_{Np'+i+1} g(x_{Np'+i+2} \dots x_{Np'+Nq'+i+2}) x_{Np'+Nq'+i+3} \dots x_{Np'+Nq'+N-k} \\ \otimes_A h(x_{Np'+Nq'+N-k+1} \dots x_{m-k+1}) x_{m-k+2} \dots x_{m+1}$$

Combining these equalities, we arrive to $E = \sum_{k=1}^{N-2} \sum_{i=0}^{N-2-k} F$, where

$$\begin{aligned} F = & f(x_1 \dots x_{Np'+1}) \dots x_{Np'+i+1} \otimes_A g(x_{Np'+i+2} \dots x_{Np'+Nq'+i+2}) \\ & \otimes_A x_{Np'+Nq'+i+3} \dots x_{Np'+Nq'+N-k} h(x_{Np'+Nq'+N-k+1} \dots x_{m-k+1}) \dots x_{m+1} \\ & - x_1 \dots x_k f(x_{k+1} \dots x_{k+Np'+1}) \dots x_{Np'+i+k+1} \otimes_A g(x_{Np'+i+k+2} \dots x_{Np'+Nq'+i+k+2}) \\ & \otimes_A x_{Np'+Nq'+i+k+3} \dots x_{Np'+Nq'+N} h(x_{Np'+Nq'+N+1} \dots x_{m+1}) \end{aligned}$$

Then, we express F as the telescopic sum $F = \sum_{\ell=0}^{k-1} G - H$, where

$$\begin{aligned} G = & x_1 \dots x_\ell f(x_{\ell+1} \dots x_{\ell+Np'+1}) \dots x_{Np'+i+1+\ell} \otimes_A g(x_{Np'+i+2+\ell} \dots x_{Np'+Nq'+i+2+\ell}) \\ & \otimes_A x_{Np'+Nq'+i+3+\ell} \dots x_{Np'+Nq'+N-k+\ell} h(x_{Np'+Nq'+N-k+1+\ell} \dots x_{m-k+1+\ell}) \dots x_{m+1}, \end{aligned}$$

$$\begin{aligned} H = & x_1 \dots x_{\ell+1} f(x_{\ell+2} \dots x_{\ell+Np'+2}) \dots x_{Np'+i+\ell+2} \otimes_A g(x_{Np'+i+\ell+3} \dots x_{Np'+Nq'+i+\ell+3}) \\ & \otimes_A x_{Np'+Nq'+i+\ell+4} \dots x_{Np'+Nq'+N-k+\ell+1} h(x_{Np'+Nq'+N-k+\ell+2} \dots x_{m-k+\ell+2}) \dots x_{m+1}. \end{aligned}$$

Finally, we conclude that, in our case 4, $as(f, g, h)$ is *always* a coboundary, by seeing that $G - H = b_K(u_\ell)$ where

$$\begin{aligned} u_\ell(x_1 \dots x_m) = & x_1 \dots x_\ell f(x_{\ell+1} \dots x_{\ell+Np'+1}) \dots x_{Np'+i+1+\ell} \otimes_A g(x_{Np'+i+2+\ell} \dots x_{Np'+Nq'+i+2+\ell}) \\ & \otimes_A x_{Np'+Nq'+i+3+\ell} \dots x_{Np'+Nq'+N-k+\ell} h(x_{Np'+Nq'+N-k+1+\ell} \dots x_{m-k+1+\ell}) \dots x_m. \end{aligned}$$

As a consequence of the previous study, we have proved the following.

Proposition 3.6 *Let $A = T(V)/(R)$ be an N -homogeneous algebra. Endowed with the Koszul cup product \smile_K , $HK^\bullet(A)$ and $HK^\bullet(A, k)$ are graded associative algebras. For any A -bimodule M , $HK^\bullet(A, M)$ is a graded $HK^\bullet(A)$ -bimodule for the actions defined by \smile_K .*

Since $HK^0(A) = Z(A)$ is the center of the algebra A , $HK^\bullet(A, M)$ is a $Z(A)$ -bimodule. From Proposition 2.5, $HK^\bullet(A, k)$ coincides with the graded algebra $W_{\nu(\bullet)}^* = \bigoplus_{p \geq 0} W_{\nu(p)}^*$ endowed with \smile_K . Definition 3.1 shows that, if $f \in W_{\nu(p)}^*$ and $g \in W_{\nu(q)}^*$, then

1. $(f \smile_K g)(x_1 \dots x_{\nu(p+q)}) = (-1)^{pq} f(x_1 \dots x_{\nu(p)}) g(x_{\nu(p)+1} \dots x_{\nu(p+q)})$ if $\nu(p+q) = \nu(p) + \nu(q)$,
2. $(f \smile_K g)(x_1 \dots x_{\nu(p+q)}) = 0$ otherwise.

In other words, if p and q are ν -additive, $f \smile_K g$ coincides up to a sign with the graded tensor product of linear forms, pre-composed with the inclusion $W_{\nu(p+q)} \hookrightarrow W_{\nu(p)} \otimes W_{\nu(q)}$.

Recall that, for any associative algebra A , $(HH^\bullet(A, k), \smile)$ is isomorphic to the Yoneda algebra $E(A) = Ext_A^*(k, k)$. If A is N -Koszul with V finite-dimensional, we know that the algebra $E(A)$ is isomorphic to $W_{\nu(\bullet)}^*$ whose product is given by the above formulas [10]. Using notation of Subsection 2.6, one has thus the following proposition.

Proposition 3.7 *Let $A = T(V)/(R)$ be an N -homogeneous algebra which is Koszul with V finite-dimensional. The map $H(\iota^*)$ is a graded algebra isomorphism from $HH^\bullet(A, k)$ to $HK^\bullet(A, k)$.*

3.3 A non-associative example at cochain level

Assume that $k = \mathbb{C}$ and that a, b and c are three complex numbers which are \mathbb{Q} -algebraically independent. We denote by A the generic AS-regular algebra of global dimension 3, cubic, of type A, defined by the parameters a, b and c [2]. Here $N = 3$. It is known that A is Koszul and Calabi-Yau [4, 10]. Recall that A is defined by two generators x and y and two cubic relations $r_1 = 0$ and $r_2 = 0$, where

$$r_1 = ay^2x + byxy + axy^2 + cx^3, \quad r_2 = ax^2y + bxyx + ayx^2 + cy^3.$$

We know that $W_4 = \mathbb{C}w$, where $w = xr_1 + yr_2$. Explicitly, one has

$$w = axy^2x + bxyxy + ax^2y^2 + cx^4 + ayx^2y + bxyx + ay^2x^2 + cy^4.$$

Keeping notation of the previous subsection, we take $M = P = Q = A$ and $p = q = r = 1$, so that f, g, h are linear maps from V to A , and we are in the case 4. Let us calculate $E = as(f, g, h)(x_1 \dots x_4)$. It is clear that $E = F$ corresponding to the unique values $k = 1$ and $i = 0$, and we obtain

$$E = f(x_1)g(x_2)h(x_3)x_4 - x_1f(x_2)g(x_3)h(x_4).$$

Let us choose $f = g$ equal to the identity of V and h constant equal to 1. Then

$$E = x_1x_2(x_4 - x_3).$$

Therefore, we arrive to

$$as(f, g, h)(w) = (a - b)(xy - yx)(x - y).$$

But $(xy - yx)(x - y)$ cannot be a linear combination of r_1 and r_2 . Thus $as(f, g, h)(w)$ is not zero in A , and the algebra $(Hom(W_{\nu(\bullet)}, A), \smile_K)$ is not associative.

It is interesting to note that this non-associativity occurs for a Koszul algebra A . As a consequence, any quasi-isomorphism from $(Hom(A^\bullet, A), b)$ to $(Hom(W_{\nu(\bullet)}, A), b_K)$ cannot send the associative usual cup product \smile to \smile_K . In fact, such a quasi-isomorphism is surjective, since it is the case if the bar resolution is replaced by the minimal projective resolution (Subsection 2.6). An example for each N of the same phenomenon including associativity of \smile_K will be presented in Section 7 (Proposition 7.11).

4 Koszul cup bracket

4.1 Definition and first properties

Definition 4.1 Let $A = T(V)/(R)$ be an N -homogeneous algebra. Let P and Q be A -bimodules, at least one of them equal to A . For any Koszul p -cochain $f : W_{\nu(p)} \rightarrow P$ and q -cochain $g : W_{\nu(q)} \rightarrow Q$, we define the Koszul cup bracket by

$$[f, g]_{\smile_K} = f \smile_K g - (-1)^{pq} g \smile_K f. \quad (4.1)$$

The Koszul cup bracket is k -bilinear, graded antisymmetric, and it passes to cohomology. We still use the notation $[\alpha, \beta]_{\smile_K}$ for the cohomology classes α and β of f and g . The Koszul cup bracket is a graded biderivation of the graded algebra $HK^\bullet(A)$. Like in the quadratic case [9], there is a remarkable 1-cocycle e_A , allowing us to relate b_K to this bracket.

Lemma 4.2 *Let $A = T(V)/(R)$ be an N -homogeneous algebra. Let $f : V \rightarrow V$ be a k -linear map, considered as 1-cochain $f : V \rightarrow A$ with coefficients in A . If f is a coboundary, then $f = 0$. If f is a cocycle, then its cohomology class contains a unique 1-cocycle with image in V and this cocycle is equal to f .*

Proof. If $f = b_K(a)$ for some a in A , then $f(x) = ax - xa$ for any x in V . Since $f(x) \in V$, this implies that $f(x) = a_0x - xa_0$ with $a_0 \in k$, thus $f = 0$. ■

If f is equal to the identity map of V , Equation (2.8) shows that for any $x_1 \dots x_N \in R$, one has $b_K(f)(x_1 \dots x_N) = Nx_1 \dots x_N$, so that $b_K(f) = 0$ since $x_1 \dots x_N$ is equal to zero in A . This Koszul 1-cocycle f is denoted by e_A and its cohomology class is denoted by \bar{e}_A . By the previous lemma, e_A is not a coboundary if $V \neq 0$. Let us call e_A the *fundamental 1-cocycle* of A , and \bar{e}_A the *fundamental 1-class* of A . From Definition 3.1, one has $e_A \smile_K e_A = 0$. The fundamental formula of the Koszul calculus [9] is generalized as follows.

Theorem 4.3 *Let $A = T(V)/(R)$ be an N -homogeneous algebra and M be an A -bimodule. For any Koszul p -cochain f with coefficients in M , we have*

1. $[e_A, f]_{\smile_K} = -b_K(f)$ if p is even,
2. $[e_A, f]_{\smile_K} = (1 - N)b_K(f)$ if p is odd.

Proof. If p is even, the proof is the same as in the quadratic case [9]. Assume that $p = 2p' + 1$. From Definition 3.1, we have

$$\begin{aligned} (e_A \smile_K f)(x_1 \dots x_{Np'+N}) \\ &= - \sum_{0 \leq j \leq N-2} (N-1-j) x_1 \dots x_{N-j-1} f(x_{N-j} \dots x_{Np'+N-j}) x_{Np'+N-j+1} \dots x_{Np'+N} \\ &= - \sum_{1 \leq i \leq N-1} i x_1 \dots x_i f(x_{i+1} \dots x_{Np'+i+1}) x_{Np'+i+2} \dots x_{Np'+N} \end{aligned}$$

which is added to

$$\begin{aligned} (f \smile_K e_A)(x_1 \dots x_{Np'+N}) \\ &= - \sum_{0 \leq i \leq N-2} (N-1-i) x_1 \dots x_i f(x_{i+1} \dots x_{Np'+i+1}) x_{Np'+i+2} \dots x_{Np'+N}, \end{aligned}$$

for obtaining

$$\begin{aligned} [e_A, f]_{\smile_K}(x_1 \dots x_{Np'+N}) &= \\ &- \sum_{0 \leq i \leq N-1} (N-1) x_1 \dots x_i f(x_{i+1} \dots x_{Np'+i+1}) x_{Np'+i+2} \dots x_{Np'+N}, \end{aligned}$$

and we conclude by (2.8). ■

Corollary 4.4 *The fundamental 1-class \bar{e}_A belongs to the graded center of $HK^\bullet(A)$, that is, for any α in $HK^\bullet(A, M)$, one has $[\bar{e}_A, \alpha]_{\underset{K}{\smile}} = 0$.*

Generalizing the quadratic case, Theorem 4.3 shows the following remarkable fact: *the Koszul differential b_K may be defined from the Koszul cup product if $N - 1$ is not divided by the characteristic of k .*

4.2 Koszul derivations

Definition 4.5 *Let $A = T(V)/(R)$ be an N -homogeneous algebra and M be an A -bimodule. Any Koszul 1-cocycle $f : V \rightarrow M$ with coefficients in M will be called a Koszul derivation of A with coefficients in M . When $M = A$, we will simply speak about a Koszul derivation of A .*

A k -linear map $f : V \rightarrow M$ is a Koszul derivation if and only if

$$\sum_{0 \leq i \leq N-1} x_1 \dots x_i f(x_{i+1}) x_{i+2} \dots x_N = 0, \quad (4.2)$$

for any $x_1 \dots x_N$ in R . If this equality holds, the unique derivation $\tilde{f} : T(V) \rightarrow M$ extending f defines a unique derivation $D_f : A \rightarrow M$ from the algebra A to the bimodule M . For example, if $f = e_A$, D_f is the *weight map* of the graded algebra A defined as in the quadratic case and denoted by D_A . The k -linear map $f \mapsto D_f$ is an isomorphism from the space of Koszul derivations of A with coefficients in M to the space of derivations from A to M . The next proposition was proved in [9] when $N = 2$.

Proposition 4.6 *Let $A = T(V)/(R)$ be an N -homogeneous algebra and M be an A -bimodule. For any Koszul derivation $f : V \rightarrow M$ and any Koszul q -cocycle $g : W_{\nu(q)} \rightarrow A$, we have*

$$[f, g]_{\underset{K}{\smile}} = b_K(D_f \circ g). \quad (4.3)$$

Proof. If q is even, the proof is similar to the proof of the quadratic case [9]. Assume that $q = 2q' + 1$. On one hand, one has

$$\begin{aligned} & (f \underset{K}{\smile} g + g \underset{K}{\smile} f)(x_1 \dots x_{Np'+N}) \\ &= - \sum_{0 \leq i+j \leq N-2} x_1 \dots x_i f(x_{i+1}) \dots x_{N-j-1} g(x_{N-j} \dots x_{Np'+N-j}) \dots x_{Np'+N} \\ &+ x_1 \dots x_i g(x_{i+1} \dots x_{Np'+i+1}) \dots x_{Np'+N-j-1} f(x_{Np'+N-j}) \dots x_{Np'+N}. \end{aligned}$$

On the other hand, we apply the derivation D_f to $b_K(g)(x_1 \dots x_{Np'+N}) = 0$ for obtaining

$$\begin{aligned} & \sum_{i=0}^{N-1} \sum_{\ell=1}^i x_1 \dots f(x_\ell) \dots g(x_{i+1} \dots x_{Np'+i+1}) \dots x_{Np'+N} \\ & + \sum_{i=0}^{N-1} x_1 \dots x_i D_f(g(x_{i+1} \dots x_{Np'+i+1})) \dots x_{Np'+N} \\ & + \sum_{i=0}^{N-1} \sum_{\ell=Np'+i+2}^{Np'+N} x_1 \dots x_i g(x_{i+1} \dots x_{Np'+i+1}) \dots f(x_\ell) \dots x_{Np'+N} = 0. \end{aligned}$$

Then the result is immediate. ■

Corollary 4.7 *Let $A = T(V)/(R)$ be an N -homogeneous algebra and M be an A -bimodule. For any $\alpha \in HK^p(A, M)$ with $p = 0$ or $p = 1$ and $\beta \in HK^q(A)$,*

$$[\alpha, \beta]_{\underset{K}{\smile}} = 0. \quad (4.4)$$

Proof. The case $p = 1$ follows from the proposition. The case $p = 0$ is clear since $HK^0(A, M)$ is the space $Z(M)$ of the elements of M commuting to any element of A . ■

We do not know whether $[\alpha, \beta]_{\underset{K}{\smile}} = 0$ holds for any p and q . A positive answer will be given for the examples of Section 7. The general case of A Koszul and $N > 2$ will be discussed after Proposition 7.12.

4.3 Higher Koszul cohomology

Let $A = T(V)/(R)$ be an N -homogeneous algebra. Let $f : V \rightarrow A$ be a Koszul derivation of A . Denote by $[f]$ the cohomology class of f . Assuming $\text{char}(k) \neq 2$, identity (4.4) shows that $[f] \underset{K}{\smile} [f] = 0$, so that the k -linear map $[f] \underset{K}{\smile} -$ defines a cochain differential on $HK^\bullet(A, M)$ for any A -bimodule M . We obtain therefore a new cohomology, called *higher Koszul cohomology* of A with coefficients in M and associated to the Koszul derivation f .

Let us limit ourselves to the case $f = e_A$, the fundamental 1-cocycle. In this case, with no assumption on the characteristic of k , the formula $\bar{e}_A \underset{K}{\smile} \bar{e}_A = 0$ and the associativity of $\underset{K}{\smile}$ on classes show that $\bar{e}_A \underset{K}{\smile} -$ is a cochain differential on $HK^\bullet(A, M)$.

Definition 4.8 *Let $A = T(V)/(R)$ be an N -homogeneous algebra and M be an A -bimodule. The differential $\bar{e}_A \underset{K}{\smile} -$ of $HK^\bullet(A, M)$ is denoted by ∂_{\smile} . The homology of $HK^\bullet(A, M)$ endowed with ∂_{\smile} is called the higher Koszul cohomology of A with coefficients in M and is denoted by $HK_{hi}^\bullet(A, M)$. We set $HK_{hi}^\bullet(A) = HK_{hi}^\bullet(A, A)$.*

If $N = 2$, we know that $e_A \underset{K}{\smile} -$ is a differential on Koszul cochains [9], but it is no longer true if $N > 2$, although $e_A \underset{K}{\smile} e_A = 0$ holds. It is due to non-associativity. Going back to the example of Subsection 3.3 and using notation of this example, we see that

$$e_A \underset{K}{\smile} (e_A \underset{K}{\smile} h) = -E$$

which is nonzero. The same example shows that $-\smile_K e_A$ does not commute with $e_A \smile_K -$ and is not a differential. With respect to this example, the following result is a bit intriguing.

Proposition 4.9 *Let $A = T(V)/(R)$ be an N -homogeneous algebra and M be an A -bimodule. The operators $e_A \smile_K -$ and $-\smile_K e_A$ are N -differentials of $\text{Hom}(W_{\nu(\bullet)}, M)$.*

Proof. Let f be a p -cochain. From

$$\begin{aligned} e_A \smile_K f(x_1 \dots x_{\nu(p+1)}) \\ &= x_1 f(x_2 \dots x_{Np'+1}) \text{ if } p = 2p' \\ &= - \sum_{i=1}^{N-1} i x_1 \dots x_i f(x_{i+1} \dots x_{Np'+i+1}) \dots x_{Np'+N} \text{ if } p = 2p' + 1. \end{aligned}$$

we deduce

$$\begin{aligned} e_A \smile_K (e_A \smile_K f)(x_1 \dots x_{\nu(p+2)}) \\ &= - \sum_{i=1}^{N-2} i x_1 \dots x_{i+1} f(x_{i+2} \dots x_{Np'+i+1}) \dots x_{Np'+N} \text{ if } p = 2p' \\ &= - \sum_{i=1}^{N-2} i x_1 \dots x_{i+1} f(x_{i+2} \dots x_{Np'+i+2}) \dots x_{Np'+N+1} \text{ if } p = 2p' + 1. \end{aligned}$$

In the two right-hand sides, the coefficients in front of f have at least two factors in V . Continuing the action of $e_A \smile_K -$, these coefficients will have successively at least three, four, ... factors, thus they vanish at the end of N actions. It is similar for the second operator. ■

Proposition 3.13 in [9] is immediately generalized as follows.

Proposition 4.10 *Given $A = T(V)/(R)$ an N -homogeneous algebra and M an A -bimodule, $HK_{hi}^0(A, M)$ is the space of the elements u of $Z(M)$ such that there exists $v \in M$ satisfying $u.x = v.x - x.v$ for any x in V . In particular, if the bimodule M is symmetric, then $HK_{hi}^0(A, M)$ is the space of elements of M annihilated by V .*

The operator $e_A \smile_K -$ vanishes if $M = k$, hence Proposition 2.5 implies that $HK_{hi}^p(A, k)$ equals $W_{\nu(p)}^*$ for any $p \geq 0$.

Proposition 4.11 *Let $A = T(V)/(R)$ be an N -homogeneous algebra. Given α in $HK^p(A)$ and β in $HK^q(A)$,*

$$\partial \smile (\alpha \smile_K \beta) = \partial \smile (\alpha) \smile_K \beta = (-1)^p \alpha \smile_K \partial \smile (\beta).$$

Proof. The same as for $N = 2$. ■

Consequently, the Koszul cup product is defined on $HK_{hi}^\bullet(A)$, still denoted by \smile_K , and $(HK_{hi}^\bullet(A), \smile_K)$ is a graded associative algebra. If $V \neq 0$, then 1 and \bar{e}_A do not survive in higher Koszul cohomology, because of $\partial \smile (1) = \bar{e}_A \neq 0$.

5 Koszul cap products

5.1 Definition and first properties

Definition 5.1 Let $A = T(V)/(R)$ be an N -homogeneous algebra. Let M and P be A -bimodules. For any p -cochain $f : W_{\nu(p)} \rightarrow P$ and any q -chain $z = m \otimes x_1 \dots x_{\nu(q)}$ in $M \otimes W_{\nu(q)}$, we define the $(q-p)$ -chains $f \underset{K}{\frown} z$ and $z \underset{K}{\frown} f$ with coefficients in $P \otimes_A M$ and $M \otimes_A P$ respectively, as follows.

1. If p and $q-p$ are not both odd, so that $\nu(q-p) = \nu(q) - \nu(p)$, one has

$$\begin{aligned} f \underset{K}{\frown} z &= (f(x_{\nu(q-p)+1} \dots x_{\nu(q)}) \otimes_A m) \otimes x_1 \dots x_{\nu(q-p)}, \\ z \underset{K}{\frown} f &= (-1)^{pq} (m \otimes_A f(x_1 \dots x_{\nu(p)})) \otimes x_{\nu(p)+1} \dots x_{\nu(q)}. \end{aligned}$$

2. If $p = 2p' + 1$ and $q = 2q'$, so that $\nu(q-p) = \nu(q) - \nu(p) - N + 2$, one has

$$\begin{aligned} f \underset{K}{\frown} z &= - \sum_{0 \leq i+j \leq N-2} (x_{Nq'-Np'-N+i+2} \dots x_{Nq'-Np'-j-1} f(x_{Nq'-Np'-j} \dots x_{Nq'-j}) \\ &\quad \otimes_A x_{Nq'-j+1} \dots x_{Nq'} m x_1 \dots x_i) \otimes x_{i+1} \dots x_{i+Nq'-Np'-N+1}. \\ z \underset{K}{\frown} f &= \sum_{0 \leq i+j \leq N-2} (x_{Nq'-j+1} \dots x_{Nq'} m x_1 \dots x_i \otimes_A f(x_{i+1} \dots x_{Np'+i+1}) \\ &\quad x_{Np'+i+2} \dots x_{Np'+N-j-1}) \otimes x_{Np'+N-j} \dots x_{Nq'-j}. \end{aligned}$$

The chain $f \underset{K}{\frown} z$ is called the left Koszul cap product of f and z , while $z \underset{K}{\frown} f$ is called their right Koszul cap product.

If $q < p$, one has $f \underset{K}{\frown} z = z \underset{K}{\frown} f = 0$. When $N = 2$, Definition 5.1 agrees definition of Koszul cap products given in [9]. When $N > 2$, the *associativity relations*

$$f \underset{K}{\frown} (g \underset{K}{\frown} z) = (f \underset{K}{\frown} g) \underset{K}{\frown} z, \quad (5.1)$$

$$(z \underset{K}{\frown} g) \underset{K}{\frown} f = z \underset{K}{\frown} (g \underset{K}{\frown} f), \quad (5.2)$$

$$f \underset{K}{\frown} (z \underset{K}{\frown} g) = (f \underset{K}{\frown} z) \underset{K}{\frown} g. \quad (5.3)$$

do not necessarily hold (in the example of Subsection 3.3, choose $f = g = e_A$ and $z = 1 \otimes r_1$). However, we will prove in the next subsection that they hold on classes.

Proposition 5.2 Let $A = T(V)/(R)$ be an N -homogeneous algebra. Let M and P be A -bimodules. For any p -cochain f with coefficients in P and any q -chain z with coefficients in M , one has

$$b_K(f \underset{K}{\frown} z) = b_K(f) \underset{K}{\frown} z + (-1)^p f \underset{K}{\frown} b_K(z), \quad (5.4)$$

$$b_K(z \underset{K}{\frown} f) = b_K(z) \underset{K}{\frown} f + (-1)^q z \underset{K}{\frown} b_K(f). \quad (5.5)$$

Proof. Let us prove only (5.4), the proof of (5.5) being similar.

1. Assume $p = 2p'$ and $q = 2q'$. From Definition 5.1 and Definition 2.6, we get

$$b_K(f \underset{K}{\frown} z) = \sum_{0 \leq i \leq N-1} (x_{i+Nq'-Np'-N+2} \dots x_{Nq'-Np'} f(x_{Nq'-Np'+1} \dots x_{Nq'}) \otimes_A m x_1 \dots x_i) \otimes x_{i+1} \dots x_{i+Nq'-Np'-N+1}, \quad (5.6)$$

$$\begin{aligned} b_K(f) \underset{K}{\frown} z = & - \sum_{0 \leq i+j \leq N-2} (x_{i+Nq'-Np'-N+2} \dots x_{Nq'-Np'-j-1} f(x_{Nq'-Np'-j} \dots x_{Nq'-j-1}) \\ & \otimes_A x_{Nq'-j} \dots x_{Nq'} m x_1 \dots x_i) \otimes x_{i+1} \dots x_{i+Nq'-Np'-N+1}, \\ & + \sum_{0 \leq i+j \leq N-2} (x_{i+Nq'-Np'-N+2} \dots x_{Nq'-Np'-j} f(x_{Nq'-Np'-j+1} \dots x_{Nq'-j}) \\ & \otimes_A x_{Nq'-j+1} \dots x_{Nq'} m x_1 \dots x_i) \otimes x_{i+1} \dots x_{i+Nq'-Np'-N+1}. \end{aligned}$$

Reducing together the two latter sums, we arrive to

$$\begin{aligned} b_K(f) \underset{K}{\frown} z = & - \sum_{0 \leq i \leq N-2} (f(x_{i+Nq'-Np'-N+2} \dots x_{i+Nq'-N+1}) \\ & \otimes_A x_{i+Nq'-N+2} \dots x_{Nq'} m x_1 \dots x_i) \otimes x_{i+1} \dots x_{i+Nq'-Np'-N+1}, \\ & + \sum_{0 \leq i \leq N-2} (x_{i+Nq'-Np'-N+2} \dots x_{Nq'-Np'} f(x_{Nq'-Np'+1} \dots x_{Nq'}) \\ & \otimes_A m x_1 \dots x_i) \otimes x_{i+1} \dots x_{i+Nq'-Np'-N+1}, \end{aligned}$$

which is added to

$$\begin{aligned} f \underset{K}{\frown} b_K(z) = & \sum_{0 \leq i \leq N-1} (f(x_{i+Nq'-Np'-N+2} \dots x_{i+Nq'-N+1}) \\ & \otimes_A x_{i+Nq'-N+2} \dots x_{Nq'} m x_1 \dots x_i) \otimes x_{i+1} \dots x_{i+Nq'-Np'-N+1}, \end{aligned}$$

for obtaining the expression of $b_K(f \underset{K}{\frown} z)$ in (5.6).

2. Assume $p = 2p'$ and $q = 2q' + 1$. We have

$$\begin{aligned} b_K(f \underset{K}{\frown} z) = & (f(x_{Nq'-Np'+2} \dots x_{Nq'+1}) \otimes_A m x_1) \otimes x_2 \dots x_{Nq'-Np'+1} \\ & - (x_{Nq'-Np'+1} f(x_{Nq'-Np'+2} \dots x_{Nq'+1}) \otimes_A m) \otimes x_1 \dots x_{Nq'-Np'}, \end{aligned}$$

$$\begin{aligned} b_K(f) \underset{K}{\frown} z = & (f(x_{Nq'-Np'+1} \dots x_{Nq'}) x_{Nq'+1} \otimes_A m) \otimes x_1 \dots x_{Nq'-Np'} \\ & - (x_{Nq'-Np'+1} f(x_{Nq'-Np'+2} \dots x_{Nq'+1}) \otimes_A m) \otimes x_1 \dots x_{Nq'-Np'}, \end{aligned}$$

$$\begin{aligned} f \underset{K}{\frown} b_K(z) = & (f(x_{Nq'-Np'+2} \dots x_{Nq'+1}) \otimes_A m x_1) \otimes x_2 \dots x_{Nq'-Np'+1} \\ & - (f(x_{Nq'-Np'+1} \dots x_{Nq'}) \otimes_A x_{Nq'+1} m) \otimes x_1 \dots x_{Nq'-Np'}. \end{aligned}$$

Then (5.4) is immediate in this case.

3. Assume $p = 2p' + 1$ and $q = 2q'$. On one hand, we have

$$\begin{aligned} b_K(f \underset{K}{\frown} z) = & - \sum_{0 \leq i+j \leq N-2} (x_{i+Nq'-Np'-N+2} \dots x_{Nq'-Np'-j-1} f(x_{Nq'-Np'-j} \dots x_{Nq'-j}) \\ & \otimes_A x_{Nq'-j+1} \dots x_{Nq'} m x_1 \dots x_{i+1}) \otimes x_{i+2} \dots x_{i+Nq'-Np'-N+1}, \\ & + \sum_{0 \leq i+j \leq N-2} (x_{i+Nq'-Np'-N+1} \dots x_{Nq'-Np'-j-1} f(x_{Nq'-Np'-j} \dots x_{Nq'-j}) \\ & \otimes_A x_{Nq'-j+1} \dots x_{Nq'} m x_1 \dots x_i) \otimes x_{i+1} \dots x_{i+Nq'-Np'-N}, \end{aligned}$$

reduced to

$$\begin{aligned} b_K(f \underset{K}{\frown} z) = & - \sum_{0 \leq i \leq N-2} (f(x_{i+Nq'-Np'-N+2} \dots x_{i+Nq'-N+2}) \\ & \otimes_A x_{i+Nq'-N+3} \dots x_{Nq'} m x_1 \dots x_{i+1}) \otimes x_{i+2} \dots x_{i+Nq'-Np'-N+1}, \\ & + \sum_{0 \leq j \leq N-2} (x_{Nq'-Np'-N+1} \dots x_{Nq'-Np'-j-1} f(x_{Nq'-Np'-j} \dots x_{Nq'-j}) \\ & \otimes_A x_{Nq'-j+1} \dots x_{Nq'} m) \otimes x_1 \dots x_{Nq'-Np'-N}. \end{aligned}$$

On the other hand, we have the two equalities

$$\begin{aligned} b_K(f) \underset{K}{\frown} z = & \sum_{0 \leq i \leq N-1} (x_{Nq'-Np'-N+1} \dots x_{Nq'-Np'-N+i} f(x_{Nq'-Np'-N+i+1} \dots x_{Nq'-N+i+1}) \\ & \otimes_A x_{Nq'-N+i+2} \dots x_{Nq'} m) \otimes x_1 \dots x_{Nq'-Np'-N}. \end{aligned}$$

$$\begin{aligned} f \underset{K}{\frown} b_K(z) = & \sum_{0 \leq i \leq N-1} (f(x_{i+Nq'-Np'-N+1} \dots x_{i+Nq'-N+1}) \\ & \otimes_A x_{i+Nq'-N+2} \dots x_{Nq'} m x_1 \dots x_i) \otimes x_{i+1} \dots x_{i+Nq'-Np'-N}, \end{aligned}$$

which are combined to obtain

$$\begin{aligned} b_K(f) \underset{K}{\frown} z - f \underset{K}{\frown} b_K(z) = & \sum_{1 \leq i \leq N-1} (x_{Nq'-Np'-N+1} \dots x_{Nq'-Np'-N+i} f(x_{Nq'-Np'-N+i+1} \dots x_{Nq'-N+i+1}) \\ & \otimes_A x_{Nq'-N+i+2} \dots x_{Nq'} m) \otimes x_1 \dots x_{Nq'-Np'-N} \\ & - \sum_{1 \leq i \leq N-1} (f(x_{i+Nq'-Np'-N+1} \dots x_{i+Nq'-N+1}) \\ & \otimes_A x_{i+Nq'-N+2} \dots x_{Nq'} m x_1 \dots x_i) \otimes x_{i+1} \dots x_{i+Nq'-Np'-N}. \end{aligned}$$

It suffices to replace i by $N-2-j$ in the first sum, and by $i+1$ in the second one, to recover $b_K(f \underset{K}{\frown} z)$.

4. Assume $p = 2p' + 1$ and $q = 2q' + 1$. In this case, one has

$$\begin{aligned} b_K(f \underset{K}{\frown} z) = & \sum_{0 \leq i \leq N-1} (x_{i+Nq'-Np'-N+2} \dots x_{Nq'-Np'} f(x_{Nq'-Np'+1} \dots x_{Nq'+1}) \\ & \otimes_A m x_1 \dots x_i) \otimes x_{i+1} \dots x_{i+Nq'-Np'-N+1}, \end{aligned} \quad (5.7)$$

$$b_K(f) \underset{K}{\frown} z = \sum_{0 \leq i \leq N-1} (x_{Nq'-Np'-N+2} \cdots x_{Nq'-Np'-N+i+1} f(x_{Nq'-Np'-N+i+2} \cdots x_{Nq'-N+i+2}) \\ \otimes_A x_{Nq'-N+i+3} \cdots x_{Nq'+1} \otimes_A m) \otimes x_1 \cdots x_{Nq'-Np'-N+1}.$$

Moreover, the following

$$f \underset{K}{\frown} b_K(z) = - \sum_{0 \leq i+j \leq N-2} (x_{i+Nq'-Np'-N+3} \cdots x_{Nq'-Np'-j} f(x_{Nq'-Np'-j+1} \cdots x_{Nq'-j+1}) \\ \otimes_A x_{Nq'-j+2} \cdots x_{Nq'+1} m x_1 \cdots x_{i+1}) \otimes x_{i+2} \cdots x_{i+Nq'-Np'-N+2}, \\ + \sum_{0 \leq i+j \leq N-2} (x_{i+Nq'-Np'-N+2} \cdots x_{Nq'-Np'-j-1} f(x_{Nq'-Np'-j} \cdots x_{Nq'-j}) \\ \otimes_A x_{Nq'-j+1} \cdots x_{Nq'+1} m x_1 \cdots x_i) \otimes x_{i+1} \cdots x_{i+Nq'-Np'-N+1}$$

is reduced to

$$f \underset{K}{\frown} b_K(z) = - \sum_{0 \leq i \leq N-2} (x_{i+Nq'-Np'-N+3} \cdots x_{Nq'-Np'} f(x_{Nq'-Np'+1} \cdots x_{Nq'+1}) \\ \otimes_A m x_1 \cdots x_{i+1}) \otimes x_{i+2} \cdots x_{i+Nq'-Np'-N+2}, \\ + \sum_{0 \leq j \leq N-2} (x_{Nq'-Np'-N+2} \cdots x_{Nq'-Np'-j-1} f(x_{Nq'-Np'-j} \cdots x_{Nq'-j}) \\ \otimes_A x_{Nq'-j+1} \cdots x_{Nq'+1} m) \otimes x_1 \cdots x_{Nq'-Np'-N+1}.$$

Replacing i by $i-1$ in the first sum, and j by $N-2-i$ in the second sum, we arrive to

$$b_K(f) \underset{K}{\frown} z - f \underset{K}{\frown} b_K(z) = \sum_{0 \leq i \leq N-1} (x_{i+Nq'-Np'-N+2} \cdots x_{Nq'-Np'} f(x_{Nq'-Np'+1} \cdots x_{Nq'+1}) \\ \otimes_A m x_1 \cdots x_i) \otimes x_{i+1} \cdots x_{i+Nq'-Np'-N+1},$$

and we recover the expression of $b_K(f \underset{K}{\frown} z)$ in (5.7). ■

Corollary 5.3 *Let $A = T(V)/(R)$ be an N -homogeneous algebra. Both Koszul cap products $\underset{K}{\frown}$ of Definition 5.1 define Koszul cap products, still denoted by $\underset{K}{\frown}$, on Koszul (co)homology classes.*

5.2 Associativity on classes

Let $A = T(V)/(R)$ be an N -homogeneous algebra with $N > 2$. Let M , P and Q be A -bimodules. For Koszul cochains $f : W_{\nu(p)} \rightarrow P$, $g : W_{\nu(q)} \rightarrow Q$ and Koszul chain $z = m \otimes x_1 \cdots x_{\nu(r)}$ in $M \otimes W_{\nu(r)}$, let us define their associators by the following Koszul $(r-p-q)$ -chains

$$as(g, f, z) = g \underset{K}{\frown} (f \underset{K}{\frown} z) - (g \underset{K}{\frown} f) \underset{K}{\frown} z, \quad (5.8)$$

$$as(z, f, g) = (z \underset{K}{\frown} f) \underset{K}{\frown} g - z \underset{K}{\frown} (f \underset{K}{\frown} g), \quad (5.9)$$

$$as(g, z, f) = g \underset{K}{\frown} (z \underset{K}{\frown} f) - (g \underset{K}{\frown} z) \underset{K}{\frown} f. \quad (5.10)$$

We assume that $r \geq p+q$ – otherwise the associators are zero.

Lemma 5.4 *One has $\nu(r - p - q) = \nu(r) - \nu(p) - \nu(q)$ if and only if at most one of the three integers $p, q, r + 1$ is odd. If at least two of these integers are odd, one has*

$$\nu(r - p - q) = \nu(r) - \nu(p) - \nu(q) - N + 2.$$

Proof. Left to the reader. ■

If $\nu(r - p - q) = \nu(r) - \nu(p) - \nu(q)$, $\underset{K}{\smile}$ and $\underset{K}{\frown}$ coincide up to a sign with \smile and \frown in all the concerned calculations, so that the three associators (5.8), (5.9), (5.10) are equal to zero. The remaining four cases are the following.

Case 1: $p = 2p', q = 2q' + 1, r = 2r'$.

Case 2: $p = 2p' + 1, q = 2q', r = 2r'$.

Case 3: $p = 2p' + 1, q = 2q' + 1, r = 2r' + 1$.

Case 4: $p = 2p' + 1, q = 2q' + 1, r = 2r'$.

Proposition 5.5 *In Case 1, $as(g, f, z) = 0$ whenever f is a cocycle.*

In Case 2, $as(g, f, z) = 0$ whenever g is a cocycle.

In Case 3, $as(g, f, z) = 0$ whenever z is a cycle.

In Case 4, $as(g, f, z)$ is a boundary.

Proof.

Case 1: $p = 2p', q = 2q' + 1, r = 2r'$. From the definitions of cup and cap products, one has

$$\begin{aligned} g \underset{K}{\frown} (f \underset{K}{\frown} z) &= - \sum_{0 \leq i+j \leq N-2} (x_{Nr'-Np'-Nq'-N+i+2} \dots x_{Nr'-Np'-Nq'-j-1} \\ &\quad g(x_{Nr'-Np'-Nq'-j} \dots x_{Nr'-Np'-j}) \otimes_A x_{Nr'-Np'-j+1} \dots x_{Nr'-Np'} \\ &\quad f(x_{Nr'-Np'+1} \dots x_{Nr'}) \otimes_A mx_1 \dots x_i) \otimes x_{i+1} \dots x_{Nr'-Np'-Nq'-N+i+1}, \\ (g \underset{K}{\smile} f) \underset{K}{\smile} z &= - \sum_{0 \leq i+j \leq N-2} (x_{Nr'-Np'-Nq'-N+i+2} \dots x_{Nr'-Np'-Nq'-j-1} \\ &\quad g(x_{Nr'-Np'-Nq'-j} \dots x_{Nr'-Np'-j}) \otimes_A f(x_{Nr'-Np'-j+1} \dots x_{Nr'-j}) \\ &\quad \otimes_A x_{Nr'-j+1} \dots x_{Nr'} mx_1 \dots x_i) \otimes x_{i+1} \dots x_{Nr'-Np'-Nq'-N+i+1}. \end{aligned}$$

Therefore we can write

$$\begin{aligned} as(g, f, z) &= \sum_{0 \leq i+j \leq N-2} (x_{Nr'-Np'-Nq'-N+i+2} \dots x_{Nr'-Np'-Nq'-j-1} \\ &\quad g(x_{Nr'-Np'-Nq'-j} \dots x_{Nr'-Np'-j}) \otimes_A E \otimes_A mx_1 \dots x_i) \otimes x_{i+1} \dots x_{Nr'-Np'-Nq'-N+i+1}, \end{aligned}$$

where E is equal to

$$f(x_{Nr'-Np'-j+1} \dots x_{Nr'-j}) \dots x_{Nr'} - x_{Nr'-Np'-j+1} \dots f(x_{Nr'-Np'+1} \dots x_{Nr'}).$$

This case is solved by writing E as the telescopic sum

$$\sum_{1 \leq \ell \leq j} x_{Nr'-Np'-j+1} \dots x_{Nr'-Np'-j+\ell-1} b_K(f)(x_{Nr'-Np'-j+\ell} \dots x_{Nr'-j+\ell}) x_{Nr'-j+\ell+1} \dots x_{Nr'}.$$

Case 2: $p = 2p' + 1$, $q = 2q'$, $r = 2r'$. One has

$$\begin{aligned} g \underset{K}{\frown} (f \underset{K}{\frown} z) &= - \sum_{0 \leq i+j \leq N-2} (g(x_{Nr'-Np'-Nq'-N+i+2} \dots x_{Nr'-Np'-N+i+1}) \\ &\quad \otimes_A (x_{Nr'-Np'-N+i+2} \dots x_{Nr'-Np'-j-1} f(x_{Nr'-Np'-j} \dots x_{Nr'-j}) \\ &\quad \otimes_A x_{Nr'-j+1} \dots x_{Nr'} m x_1 \dots x_i) \otimes x_{i+1} \dots x_{Nr'-Np'-Nq'+i+1}, \\ (g \underset{K}{\frown} f) \underset{K}{\frown} z &= - \sum_{0 \leq i+j \leq N-2} (x_{Nr'-Np'-Nq'-N+i+2} \dots x_{Nr'-Np'-Nq'-j-1} \\ &\quad g(x_{Nr'-Np'-Nq'-j} \dots x_{Nr'-Np'-j-1}) \otimes_A f(x_{Nr'-Np'-j} \dots x_{Nr'-j}) \\ &\quad \otimes_A x_{Nr'-j+1} \dots x_{Nr'} m x_1 \dots x_i) \otimes x_{i+1} \dots x_{Nr'-Np'-Nq'-N+i+1}. \end{aligned}$$

Therefore we can write

$$\begin{aligned} as(g, f, z) &= - \sum_{0 \leq i+j \leq N-2} (E \otimes_A f(x_{Nr'-Np'-j} \dots x_{Nr'-j}) \\ &\quad \otimes_A x_{Nr'-j+1} \dots x_{Nr'} m x_1 \dots x_i) \otimes x_{i+1} \dots x_{Nr'-Np'-Nq'+i+1}, \end{aligned}$$

where

$$\begin{aligned} E &= g(x_{Nr'-Np'-Nq'-N+i+2} \dots x_{Nr'-Np'-N+i+1}) \dots x_{Nr'-Np'-j-1} \\ &\quad - x_{Nr'-Np'-Nq'-N+i+2} \dots g(x_{Nr'-Np'-Nq'-j} \dots x_{Nr'-Np'-j-1}). \end{aligned}$$

Then, it suffices to write E as the telescopic sum

$$\begin{aligned} E &= \sum_{1 \leq \ell \leq N-i-j-2} x_{Nr'-Np'-Nq'-N+i+2} \dots x_{Nr'-Np'-Nq'-N+i+\ell} \\ &\quad b_K(g)(x_{Nr'-Np'-Nq'+i+\ell+1} \dots x_{Nr'-Np'+i+\ell+1}) x_{Nr'-Np'+i+\ell+2} \dots x_{Nr'-Np'-j-1}. \end{aligned}$$

Case 3: $p = 2p' + 1$, $q = 2q' + 1$, $r = 2r' + 1$. This case is technically the most difficult one. We have

$$\begin{aligned} g \underset{K}{\frown} (f \underset{K}{\frown} z) &= - \sum_{0 \leq i+j \leq N-2} (x_{Nr'-Np'-Nq'-N+i+2} \dots x_{Nr'-Np'-Nq'-j-1}) \\ &\quad g(x_{Nr'-Np'-Nq'-j} \dots x_{Nr'-Np'-j}) \otimes_A x_{Nr'-Np'-j+1} \dots x_{Nr'-Np'} \\ &\quad f(x_{Nr'-Np'+1} \dots x_{Nr'+1}) \otimes_A m x_1 \dots x_i) \otimes x_{i+1} \dots x_{Nr'-Np'-Nq'-N+i+1}, \end{aligned} \quad (5.11)$$

$$\begin{aligned} (g \underset{K}{\frown} f) \underset{K}{\frown} z &= - \sum_{0 \leq i+j \leq N-2} (x_{Nr'-Np'-Nq'-N+2} \dots x_{Nr'-Np'-Nq'-N+i+1} \\ &\quad g(x_{Nr'-Np'-Nq'-N+i+2} \dots x_{Nr'-Np'-N+i+2}) x_{Nr'-Np'-N+i+3} \dots x_{Nr'-Np'-j} \\ &\quad \otimes_A f(x_{Nr'-Np'-j+1} \dots x_{Nr'-j+1}) x_{Nr'-j+2} \dots x_{Nr'+1} \otimes_A m) x_1 \dots x_{Nr'-Np'-Nq'-N+1}. \end{aligned} \quad (5.12)$$

Let us define the linear map

$$F : M \otimes W_{\nu(r-1)} \longrightarrow (Q \otimes_A P \otimes_A M) \otimes W_{\nu(r-p-q)},$$

where $\nu(r-1) = Nr'$ and $\nu(r-p-q) = Nr' - Np' - Nq' - N + 1$, by

$$\begin{aligned} F(z') = & \sum_{0 \leq i+j+k \leq N-3} (x_{Nr'-Np'-Nq'-N+i+2} \cdots x_{Nr'-Np'-Nq'-j-k-2} \\ & g(x_{Nr'-Np'-Nq'-j-k-1} \cdots x_{Nr'-Np'-j-k-1}) \otimes_A x_{Nr'-Np'-j-k} \cdots x_{Nr'-Np'-k-1} \\ & f(x_{Nr'-Np'-k} \cdots x_{Nr'-k}) x_{Nr'-k+1} \cdots x_{Nr'} \otimes_A m x_1 \cdots x_i) \otimes x_{i+1} \cdots x_{Nr'-Np'-Nq'-N+i+1}, \end{aligned}$$

where $z' = m \otimes x_1 \cdots x_{Nr'}$. It is easy to define F intrinsically, showing that $F(z')$ does not depend on the decomposition of z' as a linear combination $m \otimes x_1 \cdots x_{Nr'}$ in $M \otimes W_{\nu(r-1)}$.

Let us choose $z' = b_K(z) = m x_1 \otimes x_2 \cdots x_{Nr'+1} - x_{Nr'+1} m \otimes x_1 \cdots x_{Nr'}$. Then $F(b_K(z))$ is the difference of the two sums

$$\begin{aligned} & \sum_{0 \leq i+j+k \leq N-3} (x_{Nr'-Np'-Nq'-N+i+3} \cdots g(x_{Nr'-Np'-Nq'-j-k} \cdots x_{Nr'-Np'-j-k}) \\ & \otimes_A x_{Nr'-Np'-j-k+1} \cdots x_{Nr'-Np'-k} f(x_{Nr'-Np'-k+1} \cdots x_{Nr'-k+1}) \\ & x_{Nr'-k+2} \cdots x_{Nr'+1} \otimes_A m x_1 \cdots x_{i+1}) \otimes x_{i+2} \cdots x_{Nr'-Np'-Nq'-N+i+2}, \\ & \sum_{0 \leq i+j+k \leq N-3} (x_{Nr'-Np'-Nq'-N+i+2} \cdots g(x_{Nr'-Np'-Nq'-j-k-1} \cdots x_{Nr'-Np'-j-k-1}) \\ & \otimes_A x_{Nr'-Np'-j-k} \cdots x_{Nr'-Np'-k-1} f(x_{Nr'-Np'-k} \cdots x_{Nr'-k}) \\ & x_{Nr'-k+1} \cdots x_{Nr'+1} \otimes_A m x_1 \cdots x_i) \otimes x_{i+1} \cdots x_{Nr'-Np'-Nq'-N+i+1}. \end{aligned}$$

If we replace i by $i+1$ and j by $j-1$ in the term of the second sum, we recover the term of the first sum, thus

$$\begin{aligned} F(b_K(z)) = & \sum_{0 \leq i+j \leq N-3} (x_{Nr'-Np'-Nq'-N+i+3} \cdots g(x_{Nr'-Np'-Nq'-j} \cdots x_{Nr'-Np'-j}) \\ & \otimes_A x_{Nr'-Np'-j+1} \cdots f(x_{Nr'-Np'+1} \cdots x_{Nr'+1}) \otimes_A m x_1 \cdots) \otimes x_{i+2} \cdots x_{Nr'-Np'-Nq'-N+i+2} \\ & - \sum_{0 \leq j+k \leq N-3} (x_{Nr'-Np'-Nq'-N+2} \cdots g(x_{Nr'-Np'-Nq'-j-k-1} \cdots x_{Nr'-Np'-j-k-1}) \\ & \otimes_A x_{Nr'-Np'-j-k} \cdots f(x_{Nr'-Np'-k} \cdots x_{Nr'-k}) \cdots x_{Nr'+1} \otimes_A m) \otimes x_1 \cdots x_{Nr'-Np'-Nq'-N+1}. \end{aligned}$$

In the latest difference, set $i' = i+1$ and $j' = j$ in the first sum, resp. $i'' = N-3-j-k$ and $j'' = j+1$ in the second sum, so that $0 \leq i' + j' \leq N-2$ with $i' \geq 1$, and $0 \leq i'' + j'' \leq N-2$ with $j'' \geq 1$. But it is easy to verify that the first sum for $i' = 0$ equals the second sum for $j'' = 0$. Therefore, $F(b_K(z))$ is equal to the difference of the right-hand sides of (5.12) and (5.11). So we obtain

$$F(b_K(z)) = -as(g, f, z),$$

which solves the Case 3.

Case 4: $p = 2p' + 1$, $q = 2q' + 1$, $r = 2r'$. One has

$$\begin{aligned}
g \underset{K}{\frown} (f \underset{K}{\frown} z) &= - \sum_{0 \leq i+j \leq N-2} (g(x_{Nr'-Np'-Nq'-N+i+1} \dots x_{Nr'-Np'-N+i+1}) \\
&\quad \otimes_A x_{Nr'-Np'-N+i+2} \dots x_{Nr'-Np'-j-1} f(x_{Nr'-Np'-j} \dots x_{Nr'-j}) \\
&\quad \otimes_A x_{Nr'-j+1} \dots x_{Nr'} m x_1 \dots x_i) \otimes x_{i+1} \dots x_{Nr'-Np'-Nq'-N+i}, \\
(g \underset{K}{\smile} f) \underset{K}{\frown} z &= - \sum_{0 \leq i+j \leq N-2} (x_{Nr'-Np'-Nq'-N+1} \dots x_{Nr'-Np'-Nq'-N+i} \\
&\quad g(x_{Nr'-Np'-Nq'-N+i+1} \dots x_{Nr'-Np'-N+i+1}) x_{Nr'-Np'-N+i+2} \dots x_{Nr'-Np'-j-1} \\
&\quad \otimes_A f(x_{Nr'-Np'-j} \dots x_{Nr'-j}) x_{Nr'-j+1} \dots x_{Nr'} \otimes_A m) \otimes x_1 \dots x_{Nr'-Np'-Nq'-N}.
\end{aligned}$$

Consequently, we are able to write

$$\begin{aligned}
as(g, f, z) &= \sum_{0 \leq i+j \leq N-2} F_{ij}, \\
F_{ij} &= (x_{Nr'-Np'-Nq'-N+1} \dots x_{Nr'-Np'-Nq'-N+i} E_{ij}) \\
&\quad \otimes x_1 \dots x_{Nr'-Np'-Nq'-N} - (E_{ij} x_1 \dots x_i) \otimes x_{i+1} \dots x_{Nr'-Np'-Nq'-N+i},
\end{aligned}$$

$$\begin{aligned}
E_{ij} &= g(x_{Nr'-Np'-Nq'-N+i+1} \dots x_{Nr'-Np'-N+i+1}) \\
&\quad \otimes_A x_{Nr'-Np'-N+i+2} \dots x_{Nr'-Np'-j-1} f(x_{Nr'-Np'-j} \dots x_{Nr'-j}) \otimes_A x_{Nr'-j+1} \dots x_{Nr'} m.
\end{aligned}$$

Writing F_{ij} as the telescopic sum

$$\begin{aligned}
F_{ij} &= \sum_{0 \leq \ell \leq i-1} x_{Nr'-Np'-Nq'-N+\ell+1} \dots x_{Nr'-Np'-Nq'-N+i} E_{ij} x_1 \dots x_\ell \otimes \dots x_{Nr'-Np'-Nq'-N+\ell} \\
&\quad - x_{Nr'-Np'-Nq'-N+\ell+2} \dots x_{Nr'-Np'-Nq'-N+i} E_{ij} x_1 \dots x_{\ell+1} \otimes \dots x_{Nr'-Np'-Nq'-N+\ell+1},
\end{aligned}$$

we see that

$$\begin{aligned}
F_{ij} &= - \sum_{0 \leq \ell \leq i-1} b_K(x_{Nr'-Np'-Nq'-N+\ell+2} \dots x_{Nr'-Np'-Nq'-N+i} \\
&\quad E_{ij} x_1 \dots x_\ell \otimes x_{\ell+1} \dots x_{Nr'-Np'-Nq'-N+\ell+1}),
\end{aligned}$$

concluding the Case 4. ■

Using similar calculations left to the reader, the statement of Proposition 5.5 holds if we replace the associator $as(g, f, z)$ by the two other ones (5.9) and (5.10).

Proposition 5.6 *Let $A = T(V)/(R)$ be an N -homogeneous algebra. Let M , P and Q be A -bimodules. For $\alpha \in HK^\bullet(A, P)$, $\beta \in HK^\bullet(A, Q)$ and $\gamma \in HK_\bullet(A, M)$, one has the associativity formulas*

$$\begin{aligned}
\beta \underset{K}{\frown} (\alpha \underset{K}{\frown} \gamma) &= (\beta \underset{K}{\smile} \alpha) \underset{K}{\frown} \gamma, \\
(\gamma \underset{K}{\frown} \alpha) \underset{K}{\frown} \beta &= \gamma \underset{K}{\frown} (\alpha \underset{K}{\smile} \beta), \\
\beta \underset{K}{\frown} (\gamma \underset{K}{\frown} \alpha) &= (\beta \underset{K}{\frown} \gamma) \underset{K}{\frown} \alpha.
\end{aligned}$$

As a consequence, $HK_\bullet(A, M)$ endowed with the actions $\widehat{}_K$ is a graded bimodule on the graded algebra $(HK^\bullet(A), \widehat{}_K)$. In particular, $HK_\bullet(A, M)$ is a $Z(A)$ -bimodule. Moreover, $HK_\bullet(A, k) = W_{\nu(\bullet)}$ is a graded bimodule on the graded algebra $HK^\bullet(A, k) = W_{\nu(\bullet)}^*$.

For $f \in W_{\nu(p)}^*$ and $z = x_1 \dots x_{\nu(q)} \in W_{\nu(q)}$, Definition 5.1 shows that

$$\begin{aligned} f \widehat{}_K z &= (-1)^{(q-p)p} f(x_{\nu(q-p)+1} \dots x_{\nu(q)}) x_1 \dots x_{\nu(q-p)}, \\ z \widehat{}_K f &= (-1)^{pq} f(x_1 \dots x_{\nu(p)}) x_{\nu(p)+1} \dots x_{\nu(q)}, \end{aligned}$$

if $\nu(q-p) = \nu(q) - \nu(p)$, and $f \widehat{}_K z = z \widehat{}_K f = 0$ otherwise.

6 Koszul cap bracket

6.1 Definition and first properties

Definition 6.1 Let $A = T(V)/(R)$ be an N -homogeneous algebra. Let M and P be A -bimodules such that M or P is equal to A . For any Koszul p -cochain $f : W_{\nu(p)} \rightarrow P$ and any Koszul q -chain $z \in M \otimes W_{\nu(q)}$, let us define the Koszul cap bracket $[f, z]_{\widehat{}_K}$ by

$$[f, z]_{\widehat{}_K} = f \widehat{}_K z - (-1)^{pq} z \widehat{}_K f. \quad (6.1)$$

The Koszul cap bracket passes to (co)homology classes. We still use the notation $[\alpha, \gamma]_{\widehat{}_K}$ for the classes α and γ of f and z respectively. When $M = A$, the map $[\alpha, -]_{\widehat{}_K}$ is a graded derivation of the graded $HK^\bullet(A)$ -bimodule $HK_\bullet(A)$.

Similarly to what happens in cohomology, the Koszul differential b_K in homology may be defined from the Koszul cap products if $N-1$ is not divided by the characteristic of k . The next theorem is analogous to Theorem 4.3. The proof is left to the reader.

Theorem 6.2 Let $A = T(V)/(R)$ be an N -homogeneous algebra and M be an A -bimodule. For any Koszul q -chain z with coefficients in M , we have

1. $[e_A, z]_{\widehat{}_K} = -b_K(z)$ if q is odd,
2. $[e_A, z]_{\widehat{}_K} = (1-N)b_K(z)$ if q is even.

Corollary 6.3 For any γ in $HK_\bullet(A, M)$, the bracket $[\bar{e}_A, \gamma]_{\widehat{}_K}$ vanishes.

6.2 Acting Koszul derivations

Using Subsection 4.2, we associate to a bimodule M and a Koszul derivation $f : V \rightarrow M$ the derivation $D_f : A \rightarrow M$. The linear map $D_f \otimes Id_{W_{\nu(\bullet)}}$ from $A \otimes W_{\nu(\bullet)}$ to $M \otimes W_{\nu(\bullet)}$ will still be denoted by D_f . The proof of the following is similar to the proof of Proposition 4.6.

Proposition 6.4 *Let $A = T(V)/(R)$ be an N -homogeneous algebra and let M be an A -bimodule. For any Koszul derivation $f : V \rightarrow M$ and any Koszul q -cycle $z \in A \otimes W_{\nu(q)}$,*

$$[f, z]_{\widehat{K}} = b_K(D_f(z)). \quad (6.2)$$

Corollary 6.5 *Let $A = T(V)/(R)$ be an N -homogeneous algebra and let M be an A -bimodule. For any $p \in \{0, 1, q\}$, $\alpha \in HK^p(A, M)$ and $\gamma \in HK_q(A)$,*

$$[\alpha, \gamma]_{\widehat{K}} = 0. \quad (6.3)$$

Proof. The case $p = 1$ follows from the proposition. The case $p = 0$ is clear. Assume that $p = q$, α is the class of f and γ is the class of $z = a \otimes x_1 \dots x_{\nu(p)}$. Definition 5.1 gives

$$[f, z]_{\widehat{K}} = f(x_1 \dots x_{\nu(p)}) \cdot a - a \cdot f(x_1 \dots x_{\nu(p)})$$

which is an element of $[M, A]_c$. Since $[\alpha, \gamma]_{\widehat{K}}$ belongs to $HK_0(A, M)$, we conclude from the isomorphism of Subsection 2.3

$$H(\tilde{\chi})_0 : HK_0(A, M) \rightarrow HH_0(A, M) = M/[M, A]_c. \blacksquare$$

Note that the same proof shows that $[\alpha, \gamma]_{\widehat{K}}$ is zero if $\alpha \in HK^p(A)$ and $\gamma \in HK_p(A, M)$. We do not know whether the identity $[\alpha, \gamma]_{\widehat{K}} = 0$ in the previous corollary holds for any p and q . A positive answer will be given for the examples of Section 7.

6.3 Higher Koszul homology

Let $A = T(V)/(R)$ be an N -homogeneous algebra. Let $f : V \rightarrow A$ be a Koszul derivation of A whose cohomology class is denoted by $[f]$. Assuming $\text{char}(k) \neq 2$, Proposition 5.6 shows that the k -linear map $[f]_{\widehat{K}} -$ defines a chain differential on $HK_{\bullet}(A, M)$ for any A -bimodule M . We obtain therefore a new homology, called *higher Koszul homology* of A with coefficients in M and associated to the Koszul derivation f . As in cohomology, if $f = e_A$, $\bar{e}_A_{\widehat{K}} -$ is a chain differential on $HK_{\bullet}(A, M)$ with no assumption on $\text{char}(k)$.

Definition 6.6 *Let $A = T(V)/(R)$ be an N -homogeneous algebra and M be an A -bimodule. The differential $\bar{e}_A_{\widehat{K}} -$ of $HK_{\bullet}(A, M)$ is denoted by ∂_{\frown} . The homology of $HK_{\bullet}(A, M)$ endowed with ∂_{\frown} is called the higher Koszul homology of A with coefficients in M and is denoted by $HK_{\bullet}^{hi}(A, M)$. We set $HK_{\bullet}^{hi}(A) = HK_{\bullet}^{hi}(A, A)$.*

If $N = 2$, we know that $e_A_{\widehat{K}} -$ is a differential on Koszul chains [9], but it is no longer true if $N > 2$. In the example of Subsection 3.3, it is easily seen that $e_A_{\widehat{K}} -$ and $- \widehat{K} e_A$ are not differentials and are not commuting. Similar calculations used in the proof of Proposition 4.9 provide the following analogue.

Proposition 6.7 *Let $A = T(V)/(R)$ be an N -homogeneous algebra and M be an A -bimodule. The operators $e_A_{\widehat{K}} -$ and $- \widehat{K} e_A$ are N -differentials of $M \otimes W_{\nu(\bullet)}$.*

The operator $e_A \underset{K}{\frown} -$ vanishes if $M = k$, hence Proposition 2.5 implies that $HK_p^{hi}(A, k) = W_{\nu(p)}$ for any $p \geq 0$.

As in the quadratic case [9], the next lemma shows that the Koszul cap products are defined for $HK_{hi}^\bullet(A)$ acting on $HK_\bullet^{hi}(A)$ – still denoted by $\underset{K}{\frown}$. More generally, $HK_\bullet^{hi}(A, M)$ is a graded bimodule on the graded algebra $HK_{hi}^\bullet(A)$.

Lemma 6.8 *Let $A = T(V)/(R)$ be an N -homogeneous algebra. Given α in $HK^p(A)$ and γ in $HK_q(A)$, one has*

$$\begin{aligned}\partial_{\frown}(\alpha \underset{K}{\frown} \gamma) &= \partial_{\frown}(\alpha) \underset{K}{\frown} \gamma = (-1)^p \alpha \underset{K}{\frown} \partial_{\frown}(\gamma), \\ \partial_{\frown}(\gamma \underset{K}{\frown} \alpha) &= \partial_{\frown}(\gamma) \underset{K}{\frown} \alpha = (-1)^q \gamma \underset{K}{\frown} \partial_{\frown}(\alpha).\end{aligned}$$

Proof. The same as for $N = 2$, using Proposition 5.6. ■

7 Koszul calculus of truncated polynomial algebras

Throughout this section, for any $N \geq 2$, let us fix $A = k[x]/(x^N)$, i.e., $A = T(V)/(R)$ where $V = k.x$ and $R = V^{\otimes N} = k.x^N$. Then A is Koszul of dimension N , generated by $1, x, \dots, x^{N-1}$. For any $p \geq 0$, $W_p = V^{\otimes p}$ is one-dimensional, generated by x^p . We assume that N is not divided by $\text{char}(k)$.

7.1 Koszul homology of A

Applying Subsection 2.2, the complex of Koszul chains of A with coefficients in A is

$$\dots \xrightarrow{b_K=0} A \otimes x^{2N} \xrightarrow{b_K} A \otimes x^{N+1} \xrightarrow{b_K=0} A \otimes x^N \xrightarrow{b_K} A \otimes x \xrightarrow{b_K=0} A \longrightarrow 0, \quad (7.1)$$

where for any a in A ,

$$\begin{aligned}b_K(a \otimes x^{Np'+1}) &= 0 \text{ if } p = 2p' + 1, \\ b_K(a \otimes x^{Np'}) &= Nx^{N-1}a \otimes x^{Np'-N+1} \text{ if } p = 2p'.\end{aligned}$$

Proposition 7.1 *The Koszul homology of A is given by*

1. $HK_0(A) = A$,
2. if $p \geq 1$ is odd, $HK_p(A)$ is $N - 1$ -dimensional, generated by the classes of $x^\ell \otimes x^{\nu(p)}$ with $0 \leq \ell \leq N - 2$,
3. if $p \geq 1$ is even, $HK_p(A)$ is $N - 1$ -dimensional, generated by the classes of $x^\ell \otimes x^{\nu(p)}$ with $1 \leq \ell \leq N - 1$.

Proof. Immediate from the complex (7.1). ■

Proposition 7.2 *The higher Koszul homology of A is given by*

1. $HK_0^{hi}(A) \cong k$,
2. $HK_p^{hi}(A) = 0$ for any $p \geq 1$.

Proof. Let us calculate $\partial_{\frown} : HK_p(A) \rightarrow HK_{p-1}(A)$ for any $p \geq 1$. If $p = 2p' + 1$ and $0 \leq \ell \leq N - 2$, one has

$$e_A \frown_K (x^\ell \otimes x^{Np'+1}) = x^{\ell+1} \otimes x^{Np'}.$$

Thus ∂_{\frown} is bijective if $p' \geq 1$, and has k as cokernel if $p' = 0$. If $p = 2p'$ and $1 \leq \ell \leq N - 1$, we easily get

$$e_A \frown_K (x^\ell \otimes x^{Np'}) = -\frac{N(N-1)}{2} x^{\ell+N-1} \otimes x^{Np'-N+1},$$

which vanishes since $x^N = 0$ in A . ■

Using Hochschild homology, it is proved in [9] that the conclusion of Proposition 7.2 holds for any Koszul A when $N = 2$ and $\text{char}(k) = 0$. It would be interesting to prove the extension to the N -case of this quadratic statement. In this kind of N -generalization, may be it is more natural to stand in the framework of N -complexes [7] or/and to use the A_∞ -structures.

Adopting a conceptual point of view, it is satisfactory to remark that the conclusion of Proposition 7.2 holds for the “point” k . By the “point” k , we mean the base field k considered as an N -homogeneous algebra with $V = 0$. The Koszul calculus of the “point” k is reduced to its 0-component which is the field k acting on vector spaces, and the higher Koszul calculus coincides with the Koszul calculus since $e_k = 0$.

7.2 Koszul cohomology of A

Applying Subsection 2.2 and using the canonical isomorphism $\text{Hom}(W_{\nu(p)}, A) \cong A \otimes W_{\nu(p)}^*$, the complex of Koszul cochains of A with coefficients in A is

$$0 \rightarrow A \xrightarrow{b_K=0} A \otimes x^* \xrightarrow{b_K} A \otimes x^{*N} \xrightarrow{b_K=0} A \otimes x^{*N+1} \xrightarrow{b_K} A \otimes x^{*2N} \rightarrow \dots, \quad (7.2)$$

where for any a in A ,

$$\begin{aligned} b_K(a \otimes x^{*Np'}) &= 0 \text{ if } p = 2p', \\ b_K(a \otimes x^{*Np'+1}) &= Nx^{N-1}a \otimes x^{*Np'+N} \text{ if } p = 2p' + 1. \end{aligned}$$

Proposition 7.3 *The Koszul cohomology of A is given by*

1. $HK^0(A) = A$,
2. if $p \geq 1$ is even, $HK^p(A)$ is $N - 1$ -dimensional, generated by the classes of $x^\ell \otimes x^{*\nu(p)}$ with $0 \leq \ell \leq N - 2$,

3. if $p \geq 1$ is odd, $HK^p(A)$ is $N - 1$ -dimensional, generated by the classes of $x^\ell \otimes x^{*\nu(p)}$ with $1 \leq \ell \leq N - 1$.

Proof. Immediate from the complex (7.2). ■

Proposition 7.4 *The higher Koszul cohomology of A is given by*

1. $HK_{hi}^0(A) \cong kx^{N-1}$,
2. $HK_{hi}^p(A) = 0$ for any $p \geq 1$.

Proof. Let us calculate $\partial_\cup : HK^p(A) \rightarrow HK^{p-1}(A)$ for any $p \geq 0$. If $p = 2p'$ and $0 \leq \ell \leq N - 1$, one has

$$e_A \smile_K (x^\ell \otimes x^{*Np'}) = x^{\ell+1} \otimes x^{*Np'+1}.$$

Thus ∂_\cup is bijective if $p' \geq 1$, and has kx^{N-1} as kernel if $p' = 0$. If $p = 2p' + 1$ and $1 \leq \ell \leq N - 1$, we get

$$e_A \smile_K (x^\ell \otimes x^{*Np'}) = -\frac{N(N-1)}{2} x^{\ell+N-1} \otimes x^{*Np'+N},$$

vanishing again. ■

It is conjectured in [9] that a Koszul quadratic A such that there exists n satisfying $HK_{hi}^p(A) = 0$ for any $p \neq n$ and $HK_{hi}^n(A) \cong k$, is n -Calabi-Yau. Clearly, our example $A = k[x]/(x^N)$ is not 0-Calabi-Yau, so that the conjecture fails when $N = 2$. Let us show how improving the conjecture in order to include our example. Actually, as proved below, the product on $HK_{hi}^0(A) \cong kx^{N-1}$ is zero. Then it suffices to impose that the isomorphism

$$HK_{hi}^\bullet(A) \cong k$$

is an isomorphism of *graded algebras*, where k is the graded algebra concentrated in degree n , and its product is the product of k if $n = 0$, otherwise it is zero. Moreover, the “point” k is 0-Calabi-Yau and satisfies the improved conjecture.

7.3 Koszul cup product of A

Proposition 7.5 *For $f : kx^{\nu(p)} \rightarrow A$ and $g : kx^{\nu(q)} \rightarrow A$, the map $f \smile_K g : kx^{\nu(p+q)} \rightarrow A$ is given by*

1. $f \smile_K g(x^{\nu(p+q)}) = f(x^{\nu(p)})g(x^{\nu(q)})$ if p and q are not both odd,
2. $f \smile_K g(x^{\nu(p+q)}) = -\frac{N(N-1)}{2} x^{N-2} f(x^{\nu(p)})g(x^{\nu(q)})$ if p and q are both odd.

Proof. We may assume that $f(x^{\nu(p)}) = x^i$ and $g(x^{\nu(q)}) = x^j$, where $0 \leq i, j \leq N - 1$. If p and q are not both odd, then $f \smile_K g(x^{\nu(p+q)}) = x^{i+j}$. If $p = 2p' + 1$ and $q = 2q' + 1$, then

$$f \smile_K g(x^{Np'+Nq'+N}) = - \sum_{0 \leq i'+j' \leq N-2} x^{i'} x^i x^{N-2-i'-j'} x^j x^{j'},$$

which is equal to $-\frac{N(N-1)}{2} x^{N-2+i+j}$. ■

Corollary 7.6 1. The Koszul cup product of $\text{Hom}(W_{\nu(p)}, A)$ is associative and commutative (but not graded commutative if $\text{char}(k) \neq 2$).

2. The Koszul cup product of $HK^\bullet(A)$ is commutative and graded commutative.

3. The Koszul cup product of $HK_{hi}^\bullet(A)$ is zero.

Proof. 1. Associativity is easily verified from the formulas of the proposition. Commutativity is clear from these formulas since A is commutative. If $p = 2p' + 1$ and $q = 2q' + 1$, then $f \underset{K}{\smile} g \neq -g \underset{K}{\smile} f$ whenever $i + j \leq 1$ and $\text{char}(k) \neq 2$.

2. Graded commutativity comes from the formula $[f] \underset{K}{\smile} [g] = 0$ if $p = 2p' + 1$ and $q = 2q' + 1$.

3. Proposition 7.4 shows that $HK_{hi}^\bullet(A)$ is reduced to $HK_{hi}^0(A) \cong kx^{N-1}$, and one has $x^{N-1} \underset{K}{\smile} x^{N-1} = 0$. ■

7.4 Koszul cap products of A

Proposition 7.7 For $f : kx^{\nu(p)} \rightarrow A$ and $z = z' \otimes x^{\nu(q)}$ where $z' \in A$ and $q \geq p$, the elements $f \underset{K}{\frown} z$ and $z \underset{K}{\frown} f$ of $A \otimes x^{\nu(q-p)}$ are given by

1. $f \underset{K}{\frown} z = (-1)^{pq} z \underset{K}{\frown} f = f(x^{\nu(p)}) z' \otimes x^{\nu(q-p)}$ if p and $q - p$ are not both odd,

2. $f \underset{K}{\frown} z = -z \underset{K}{\frown} f = -\frac{N(N-1)}{2} x^{N-2} f(x^{\nu(p)}) z' \otimes x^{\nu(q-p)}$ if p odd and q even.

Proof. We may assume that $f(x^{\nu(p)}) = x^i$ and $z' = x^j$, where $0 \leq i, j \leq N - 1$. If p and $q - p$ are not both odd, then $f \underset{K}{\frown} z = (-1)^{pq} z \underset{K}{\frown} f = x^{i+j} \otimes x^{\nu(q-p)}$. If $p = 2p' + 1$ and $q = 2q'$, then

$$f \underset{K}{\frown} z = -z \underset{K}{\frown} f = - \sum_{0 \leq i' + j' \leq N-2} x^{j'} x^i x^{N-2-i'-j'} x^j x^{i'} \otimes x^{Nq' - Np' - N+1}$$

which is equal to $-\frac{N(N-1)}{2} x^{N-2+i+j} \otimes x^{\nu(q-p)}$. ■

Corollary 7.8 1. Koszul cup and cap products satisfy the associativity relations (5.1)-(5.3) on chains-cochains. In particular, $A \otimes W_{\nu(p)}$ is a graded $\text{Hom}(W_{\nu(p)}, A)$ -bimodule. If $\text{char}(k) \neq 2$, this bimodule is neither symmetric, nor graded symmetric.

2. The $HK^\bullet(A)$ -bimodule $HK_\bullet(A)$ is graded symmetric. If $\text{char}(k) \neq 2$, this bimodule is not symmetric.

3. The Koszul cap products of $HK_{hi}^\bullet(A)$ acting on $HK_{hi}^\bullet(A)$ are zero.

Proof. 1. The rather long calculations leading to associativity relations from the formulas of the proposition are routine and are left to the reader. If $p = 2p' + 1$, $q = 2q'$, then $f \underset{K}{\frown} z = -z \underset{K}{\frown} f$ is not zero whenever $i + j \leq 1$, so the bimodule is neither symmetric, nor graded symmetric when $\text{char}(k) \neq 2$.

2. Graded symmetry comes from the formula $[f] \underset{K}{\frown} [z] = 0$ if $p = 2p' + 1$ and $q = 2q'$. Non-symmetry occurs from the case p and q odd.
3. Proposition 7.2 shows that $HK_{\bullet}^{hi}(A)$ is reduced to $HK_0^{hi}(A) \cong k$, on which $HK_{hi}^0(A) \cong kx^{N-1}$ acts by zero. ■

7.5 A comparison morphism for A

Keeping notation of Subsection 2.3, our aim is to complete the construction of the morphism of complexes $\chi : K(A) \rightarrow \bar{B}(A)$ from the extra degeneracy s , when $A = k[x]/(x^N)$. Since A is Koszul, χ is a comparison morphism, i.e., a morphism between the two resolutions $K(A)$ and $\bar{B}(A)$ of A . We have already obtained that

$$\chi_2(a \otimes x^N \otimes a') = \sum_{1 \leq i \leq N-1} a \otimes x^i \otimes x \otimes x^{N-1-i} a'.$$

From $s_2(a \otimes a_1 \otimes a_2 \otimes a') = 1 \otimes \bar{a} \otimes a_1 \otimes a_2 \otimes a'$, we deduce

$$\begin{aligned} \chi_3(1 \otimes x^{N+1} \otimes 1) &= (s_2 \circ \chi_2 \circ d)(1 \otimes x^{N+1} \otimes 1) \\ &= (s_2 \circ \chi_2)(x \otimes x^N \otimes 1 - 1 \otimes x^N \otimes x) \\ &= s_2\left(\sum_{1 \leq i \leq N-1} x \otimes x^i \otimes x \otimes x^{N-1-i} - 1 \otimes x^i \otimes x \otimes x^{N-1-i}\right) \\ &= \sum_{1 \leq i \leq N-1} 1 \otimes x \otimes x^i \otimes x \otimes x^{N-1-i}, \end{aligned}$$

obtaining

$$\chi_3(a \otimes x^{N+1} \otimes a') = \sum_{1 \leq i \leq N-1} a \otimes x \otimes x^i \otimes x \otimes x^{N-1-i} a'.$$

Similarly, one has

$$\begin{aligned} \chi_4(1 \otimes x^{2N} \otimes 1) &= (s_3 \circ \chi_3 \circ d)(1 \otimes x^{2N} \otimes 1) \\ &= (s_3 \circ \chi_3)\left(\sum_{0 \leq j \leq N-1} x^j \otimes x^{N+1} \otimes x^{N-1-j}\right) \\ &= s_3\left(\sum_{1 \leq i \leq N-1, 0 \leq j \leq N-1} x^j \otimes x \otimes x^i \otimes x \otimes x^{2N-2-i-j}\right) \\ &= \sum_{1 \leq i, j \leq N-1} 1 \otimes x^j \otimes x \otimes x^i \otimes x \otimes x^{2N-2-i-j}, \end{aligned}$$

and we may impose that $i + j \geq N - 1$ in the sum. We have obtained

$$\chi_4(a \otimes x^{2N} \otimes a') = \sum_{1 \leq i, j \leq N-1} a \otimes x^j \otimes x \otimes x^i \otimes x \otimes x^{2N-2-i-j} a'.$$

Next proposition is an immediate generalization of the obtained formulas.

Proposition 7.9 *The resolution morphism $\chi : K(A) \rightarrow \bar{B}(A)$ defined by the extra degeneracy s is given by*

$$\begin{aligned}\chi_{2p'}(a \otimes x^{Np'} \otimes a') &= \sum_{1 \leq i_1, \dots, i_{p'} \leq N-1} a \otimes x^{i_{p'}} \otimes x \dots x \otimes x^{i_1} \otimes x \otimes x^{(N-1)p' - i_1 - \dots - i_{p'}} a', \\ \chi_{2p'+1}(a \otimes x^{Np'+1} \otimes a') &= \sum_{1 \leq i_1, \dots, i_{p'} \leq N-1} a \otimes x \otimes x^{i_{p'}} \otimes x \dots x \otimes x^{i_1} \otimes x \otimes x^{(N-1)p' - i_1 - \dots - i_{p'}} a',\end{aligned}$$

and we may impose that $i_1 + \dots + i_{p'} \geq (N-1)(p'-1)$ in these sums. If $N = 2$, χ coincides with the inclusion of $K(A)$ into $\bar{B}(A)$.

Corollary 7.10 *For any A -bimodule M , the quasi-isomorphisms $\tilde{\chi} : M \otimes W_{\nu(\bullet)} \rightarrow M \otimes \bar{A}^{\otimes \bullet}$ and $\chi^* : \text{Hom}(\bar{A}^{\otimes \bullet}, M) \rightarrow \text{Hom}(W_{\nu(\bullet)}, M)$ deduced from χ are given by*

$$\begin{aligned}\tilde{\chi}_{2p'}(m \otimes x^{Np'}) &= \sum_{1 \leq i_1, \dots, i_{p'} \leq N-1} x^{(N-1)p' - i_1 - \dots - i_{p'}} m \otimes (x^{i_{p'}} \otimes x \dots x \otimes x^{i_1} \otimes x), \\ \tilde{\chi}_{2p'+1}(m \otimes x^{Np'+1}) &= \sum_{1 \leq i_1, \dots, i_{p'} \leq N-1} x^{(N-1)p' - i_1 - \dots - i_{p'}} m \otimes (x \otimes x^{i_{p'}} \otimes x \dots x \otimes x^{i_1} \otimes x),\end{aligned}$$

and, for $f : \bar{A}^{\otimes p} \rightarrow M$, by

$$\begin{aligned}\chi_{2p'}^*(f)(x^{Np'}) &= \sum_{1 \leq i_1, \dots, i_{p'} \leq N-1} f(x^{i_{p'}} \otimes x \dots x \otimes x^{i_1} \otimes x) x^{(N-1)p' - i_1 - \dots - i_{p'}}, \\ \chi_{2p'+1}^*(f)(x^{Np'+1}) &= \sum_{1 \leq i_1, \dots, i_{p'} \leq N-1} f(x \otimes x^{i_{p'}} \otimes x \dots x \otimes x^{i_1} \otimes x) x^{(N-1)p' - i_1 - \dots - i_{p'}}.\end{aligned}$$

We may impose that $i_1 + \dots + i_{p'} \geq (N-1)(p'-1)$ in these sums.

Proposition 7.11 *For $N > 2$ and $M = A$, χ^* is not an algebra morphism, and $\tilde{\chi}$ is not a bimodule morphism.*

Proof. Let $f : \bar{A} \rightarrow A$ be the 1-cochain defined by $f(x^i) = 0$ if $i \neq 2$ and $f(x^2) = 1$. Then $\chi^*(f) = 0$, thus $\chi^*(f) \smile_K \chi^*(D_A) = 0$. But $f \smile D_A : \bar{A} \otimes \bar{A} \rightarrow A$ is defined by $f \smile D_A(a \otimes a') = -f(a)D_A(a')$, so that $\chi^*(f \smile D_A) : W_N \rightarrow A$ is given by

$$\chi^*(f \smile D_A)(x^N) = \sum_{1 \leq i \leq N-1} f \smile D_A(x^i \otimes x) x^{N-1-i} = -x^{N-2},$$

which is not zero.

Similarly, if z is the Koszul 2-cycle defined by $z = x \otimes x^N$, one has $\tilde{\chi}(z \smile_K \chi^*(f)) = 0$, while

$$\tilde{\chi}(z) = \sum_{1 \leq i \leq N-1} x^{N-i} \otimes (x^i \otimes x),$$

implying

$$\tilde{\chi}(z) \smile f = \sum_{1 \leq i \leq N-1} x^{N-i} f(x^i) \otimes x = x^{N-2} \otimes x,$$

which is not zero. ■

Proposition 7.12 *For $N > 2$, $H(\chi^*) : HH^\bullet(A) \rightarrow HK^\bullet(A)$ is an algebra isomorphism, and $H(\tilde{\chi}) : HK_\bullet(A) \rightarrow HH_\bullet(A)$ is a bimodule isomorphism.*

Proof. Since A is Koszul, $H(\chi^*)$ and $H(\tilde{\chi})$ are linear isomorphisms. We will use the basis $f_{m,i}$ of Hochschild cocycles given at the end of the paper [1]. It is clear that the image by χ^* of this basis is exactly the basis of Koszul cocycles given in Proposition 7.3. Moreover the formulas of [1] giving the cup product between the $f_{m,i}$'s agree the formulas of Proposition 7.5, showing that $H(\chi^*)$ is an algebra morphism.

It is easy to provide the image by $\tilde{\chi}$ of the basis of Koszul cycles of Proposition 7.1, obtaining a basis of Hochschild cycles. Here again, the formulas giving the actions of the $f_{m,i}$'s on these Hochschild cycles agree the formulas of Proposition 7.7, showing that $H(\tilde{\chi})$ is a bimodule morphism. ■

We do not know whether the statement of Proposition 7.12 holds for any N -Koszul algebra A , where χ is still the comparison morphism defined by the extra degeneracy s . In case of a positive answer, the algebra $HK^\bullet(A)$ would be graded commutative for any N -Koszul algebra A , since the algebra $HH^\bullet(A)$ is graded commutative [20]. When A is any confluent N -Koszul algebra, it is possible to make explicit a comparison morphism in the opposite direction, i.e., from $\bar{B}(A)$ to $K(A)$, by following the same method used for the construction of χ from a contracting homotopy of $\bar{B}(A)$. Actually, in this situation, an explicit contracting homotopy of $K(A)$ has been recently defined by Chenavier [13].

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